Calculating $\mathcal{M}$ from Feynman diagrams

claim: S-matrix is "simply the Fourier transform of an n-point correlation function"

⇝ "LSZ reduction formula"

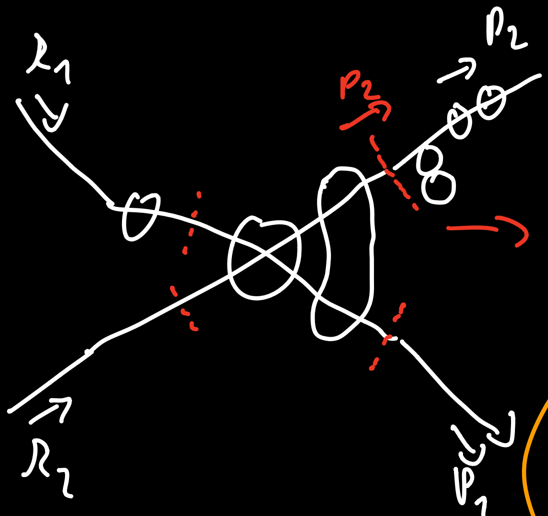
[Lehman, Symanzik & Zimmermann
proof → QFT 2!]

$$\prod_{i=1}^{n}\int d^4x_i\, e^{ip_ix_i}\prod_{j=1}^{m}\int d^4y_j\, e^{-ik_jy_j}\langle\Omega|T\{\overbrace{\phi(x_1)\dots\phi(x_n)}^{\text{"out"}}\overbrace{\phi(y_1)\dots\phi(y_m)}^{\text{"in"}}\}|\Omega\rangle$$

$$\xrightarrow[\substack{p_i^0\to E_{\vec p_i}\\ k_i^0\to E_{\vec k_i}}]{}\left(\prod_{i=1}^{n}\frac{\sqrt{Z}\,i}{p_i^2-m^2+i\epsilon}\right)\left(\prod_{j=1}^{m}\frac{\sqrt{Z}\,i}{k_j^2-m^2+i\epsilon}\right)\langle\vec p_1\dots\vec p_n|S|\vec k_1\dots k_m\rangle$$

(@ any common reference time)

consider an individual diagram [in ϕ^4 theory]



$\to \sim \left.\dfrac{i}{p_2^2-m^2}\right|_{p_2^2=m^2}$

~ describes $|\vec p\rangle_0 \to |\vec p\rangle$

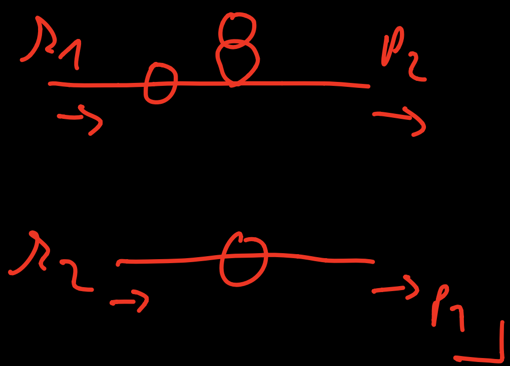
⇝ nothing to do with

actual scattering process

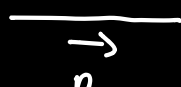
=> $\boxed{i\mathcal{M} = \text{sum of all fully connected, amputated diagrams}}$

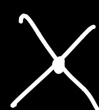
$S = 1 + iT$

⌈i.e. do not include, e.g.,



⌋

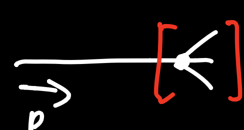
→ rules ⌈for $i\mathcal{M}$!⌋ : 1. propagator  $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

2. vertex  $= -i\lambda$

& impose 4-momentum conservation @ each vertex

& integrate over each undetermined (=loop!) momenta

3. divide by symmetry factor

! 4. external lines  $= 1$

[$\Leftrightarrow$ poles for correlation functions!]

Motivation for LSZ:

$$\langle \vec{p}_1 \vec{p}_2 \dots | S | \vec{k}_A \vec{k}_B \rangle = {}_{out}\langle \vec{p}_1 \vec{p}_2 \dots | k_A k_B \rangle_{in}$$

$$= \lim_{T\to\infty} \langle \vec{p}_1 \vec{p}_2 \dots | e^{-iH(2T)} | \vec{k}_A \vec{k}_B \rangle$$

recall: $|\Omega\rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT} |0\rangle$

$$\text{"}\Rightarrow\text{"} \quad |k_A k_B\rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT} |k_A k_B\rangle_0$$

$$\Rightarrow \langle \vec{p}_1 \vec{p}_2 \dots | S | \vec{k}_A \vec{k}_B \rangle \propto \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \vec{p}_1 \vec{p}_2 | T\left\{ \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\} | \vec{k}_A \vec{k}_B \rangle_0$$

Summary "QFT in a nutshell"

$$\boxed{\begin{aligned}\langle \vec{p}_1 \vec{p}_2 \ldots | iT | \vec{k}_A \vec{k}_B \rangle &\equiv i\mathcal{M} (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_f) \\ &= {}_0\langle \vec{p}_1 \vec{p}_2 \ldots | T\{\exp[-i\int_{-T}^{T} dt\, H_I(t)]\} | \vec{k}_A \vec{k}_B \rangle_0 \Big|_{\#}\end{aligned}}$$

fully connected + amputated

1. Expand exp[..] in coupling constant(s)
2. Use Wick's theorem to expand T{...}
3. contract every external state w/ one operator from the expansion
4. Contract all remaining operator w/ each other
5. Disregard amplitudes that can be "amputated"

[= any of the propagators is on shell]

example: $\mathcal{O}(\lambda)$ contribution to $\langle \vec{p}_1 \vec{p}_2 | iT | \vec{k}_A \vec{k}_B \rangle$

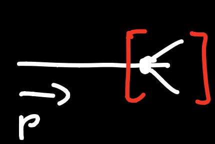
$$\rightarrow {}_0\langle \vec{p}_1 \vec{p}_2 | -i \frac{\lambda}{4!} T\{\int d^4x\, \phi_I(x)^4\} | \vec{k}_A \vec{k}_B \rangle$$

$$= -i \frac{\lambda}{4!} \int d^4x\, {}_0\langle \vec{p}_1 \vec{p}_2 | \mathcal{N}\{\phi(x)\phi(x)\phi(x)\phi(x) + \text{all possible contractions}\} | \vec{k}_A \vec{k}_B \rangle$$

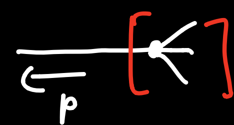
$$\text{e.g. } \phi^+(x) | \vec{p} \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} a_k e^{ikx} \sqrt{2E_p}\, a_p^+ |0\rangle$$

$$= e^{-ipx} |0\rangle$$

$\Rightarrow$ $\overline{\phi(x)\,|\vec{p}\rangle}_0 \equiv e^{-ipx}|0\rangle$

$\hat{=}$  $= 1$

${}_0\langle \overline{\vec{p}\,|\,\phi}(x) \equiv {}_0\langle 0|\, e^{+ipx}$

$\hat{=}$  $= 1$

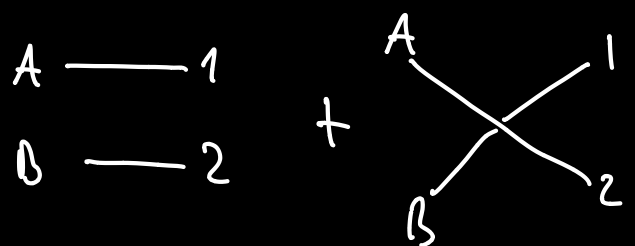
NB: In total, only equal numbers of $a^\dagger$ and a survive

in $\langle \vec{p}_1 \vec{p}_2 \dots |\, \phi^m \,| \vec{k}_A \vec{k}_B \rangle \sim \langle 0|(a)^n (\phi^+ + \phi^-)^m (a^\dagger)^2|0\rangle$

$\rightsquigarrow$ every ϕ must be "contracted" with either initial or final state!

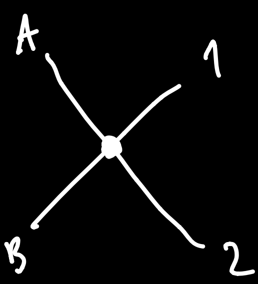
$\rightsquigarrow$ consider all possible full contractions of ϕ and external state momenta!

e.g. $\bullet\; -i\frac{\lambda}{4!}\int d^4x\; {}_0\langle \vec{p}_1 \vec{p}_2 \,|\, \overline{\phi\,\phi}\;\overline{\phi\,\phi} \,|\, \vec{k}_A \vec{k}_B \rangle_0$ $(\cdot\, 3\times 2)$

$= 8 \times \Big($  $\Big)$

part of the "1" in $S = 1 + iT$

$\rightsquigarrow$ ignore



- $-i\frac{\lambda}{4!}\int d^4x \; {}_0\langle \vec{p}_1 \vec{p}_2 | \phi\phi\phi\phi | \vec{k}_A \vec{k}_B \rangle$ (4! options)

$$= 4!\left(-i\frac{\lambda}{4!}\right)\int d^4x \, e^{+ip_1x} e^{+ip_2x} e^{-ik_Ax} e^{-ik_Bx} \underbrace{\langle 0|0\rangle}_{1}$$

$$= \underbrace{-i\lambda}_{=i\mathcal{M}} (2\pi)^4 \delta^{(4)}(p_1+p_2-k_A-k_B)$$

✓ when directly applying Feynman rules!

7. Feynman rules for fermions

Wick's theorem

$$\boxed{T\{\psi_1\bar\psi_2\psi_3\ldots\} = N\{\psi_1\bar\psi_2\psi_3\ldots + \text{all possible contractions}\}}$$

where • $\overbracket{\psi(x)\bar\psi}(y) \equiv \begin{cases}\{\psi^+(x),\bar\psi^-(x)\} & \text{for } x^0>y^0 \\ -\{\bar\psi^+(x),\psi^-(x)\} & \text{for } x^0<y^0\end{cases} = S_F(x-y)$

• $\overbracket{\psi\psi} = \overbracket{\bar\psi\bar\psi} = 0$

• $N(\overbracket{\psi_1\psi_2\bar\psi_3}\bar\psi_4) = -N(\overbracket{\psi_1\bar\psi_3}\psi_2\bar\psi_4)$

$= -\overbracket{\psi_1\bar\psi_3}\,N(\psi_2\bar\psi_4)$

etc.

contractions with external states

e.g. $\psi^+(x)|\vec p,s\rangle_{fermion} = \int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'}u^{s'}(p')a^{s'}_{p'}e^{-ip'x}\sqrt{2E_p}\,a_p^{s\dagger}|0\rangle$

$= e^{-ipx}u^s(p)|0\rangle$

$\equiv \overbracket{\psi_I(x)|\vec p},s\rangle_{fermion}$

similarly: $\overbracket{\bar\psi(x)|\vec p},s\rangle_{anti\text{-}fermion} = e^{-ipx}\bar v^s(p)|0\rangle$

Yukawa theory

$$H = H_{Dirac} + H_{Klein\text{-}Gordon} + H_I$$

$$H_I = g\int d^3x\, \bar{\psi}\psi\phi$$

<u>example</u>: consider fermion (k) + fermion (p) $\longrightarrow$ fermion (k') + fermion (p')

$\rightarrow$ leading contribution to S-matrix from H_I^2 term:

$$_0\langle \vec{p}'\vec{k}'|T\{\tfrac{1}{2!}(-ig)^2\int d^4x\int d^4y\,(\bar{\psi}_a\psi_a\phi)_x\,(\bar{\psi}\psi\phi)_y\}|\vec{k}\,\vec{p}\rangle_0$$

$\rightsquigarrow$ 2x2 contractions possible

$$\supset (-ig)^2\int d^4x\int d^4y\,(\bar{u}_a(p')u_a(p))(\bar{u}(k')u(k))\times$$

$$\times\, e^{+ik'y}e^{+ip'x}e^{-iky}e^{-ipx}$$

$$\times\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\varepsilon}e^{-iq(x-y)}$$

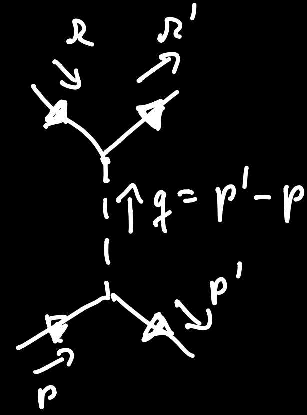
$$= -ig^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m_\phi^2+i\varepsilon}(2\pi)^8\,\delta^{(4)}(p'-p-q)\,\underbrace{\delta^{(4)}(k'-k+q)}_{\Rightarrow\,\delta^{(4)}(k'-k+p'-p)}$$

$\Rightarrow q = p'-p$

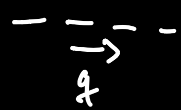
$$\times\,(\bar{u}_a(p')u_a(p))(\bar{u}(k')u(k))$$

$$= \underbrace{\frac{-ig^2}{(p'-p)^2-m_\phi^2}\left(\bar u_a(p')\,u_a(p)\right)\left(\bar u(k')\,u(k)\right)}_{= i\mathcal{M}} \cdot (2\pi)^4\,\delta^{(4)}(p+k-p'-k')$$

$\Longleftrightarrow$ with Feynman rules:

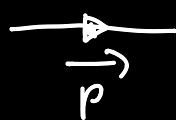


1. propagators $\phi\,\phi$



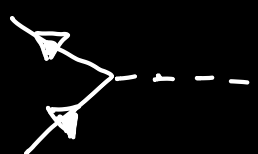
$$= \frac{i}{q^2 - m_\phi^2 + i\varepsilon}$$

$\psi\,\bar\psi$



$$= \frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}$$

2. vertices

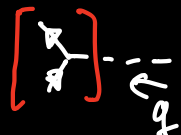
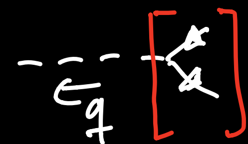
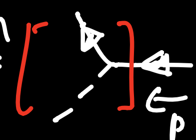
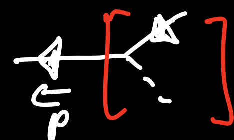
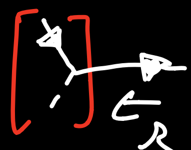
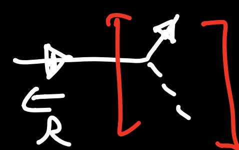


$$= -ig$$

& impose 4-momentum conservation @ every vertex

& integrate over all undetermined (loop) momenta ⌈keep "$+i\varepsilon$" only for this case!⌋

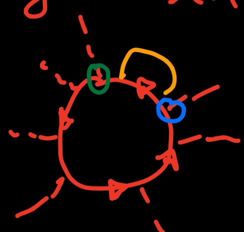
3. external leg contractions:

ingoing particles	outgoing particles
$\phi \| \vec{q}\rangle \;\hat{=}\;$  $= 1$	$\langle \vec{q} \| \phi^{+} \;\hat{=}\;$  $= 1$
$\psi \| \vec{p}, s\rangle_{\text{fermion}} \;\hat{=}\;$  $= u^s(p)$	$\langle \vec{p}, s \| \bar{\psi} \;\hat{=}\;$  $= \bar{u}^s(p)$
$\bar{\psi} \| \vec{k}, s\rangle_{\text{anti-fermion}} \;\hat{=}\;$  $= \bar{v}^s(k)$	$\langle \vec{k}, s \| \psi \;\hat{=}\;$  $= v^s(p)$

4. overall sign?

→ determined by anti-commutation of ψ!

e.g. closed fermion loop:



$$= (\bar{\psi}_a \psi_a)_x (\bar{\psi}_b \psi_b)_y \dots (\bar{\psi}_d \psi_d)$$

$$= -\psi_d \bar{\psi}_a \;\psi_a \bar{\psi}_b \;\dots\; \psi_b \bar{\psi}_d$$

$$= -\; S_{da} S_{ab} \dots S_{bd} = -\mathrm{Tr}[S \dots S]$$

⇝ 1 overall minus sign per fermion loop

5. <u>no</u> symmetry factors!

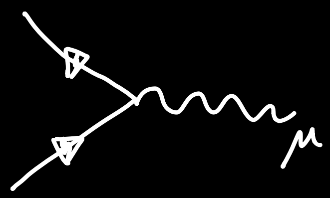
because H_I has 3 <u>different</u> term

Quantum Electrodynamics (QED)

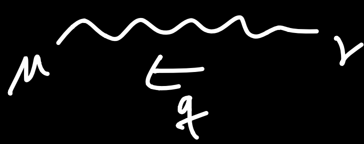
$$H_{int} = \int d^3x \; g \bar{\psi}\psi\phi \rightarrow \int d^3x \; e \bar{\psi}\gamma^\mu\psi A_\mu = \int d^3x \; e\bar{\psi}\not{A}\psi$$

$\Rightarrow$

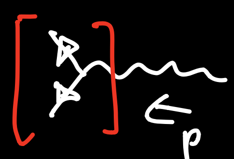
⌜proof later⌟

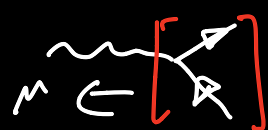


$= -ie\gamma^\mu \quad (= -iQ|e|\gamma^\mu)$



$= \dfrac{-ig^{\mu\nu}}{q^2 + i\varepsilon}$

$\langle 0 | A_\mu | \vec{p}\rangle \;\hat{=}$  $= \varepsilon_\mu(p)$ "polarization vector"

$\langle \vec{p} | A_\mu | 0 \rangle \;\hat{=}$  $= \varepsilon^*_\mu(p)$

w/ $\vec{p}\cdot\vec{\varepsilon} = 0$

⌜$p\cdot\varepsilon = 0$ from e.o.m., only for massless particles

e.g. $\varepsilon^{\pm} = (0, 1, \pm i, 0)\cdot\frac{1}{\sqrt{2}}$

for circular polarization