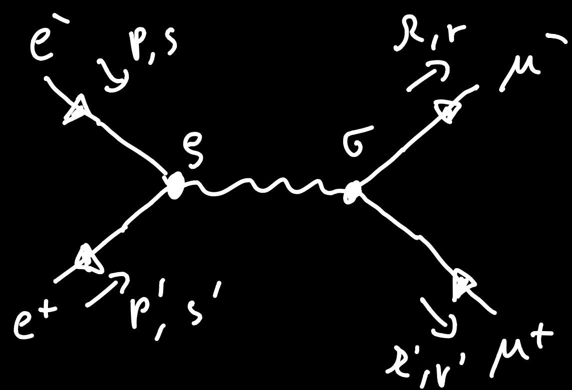
8. Elementary QED processes

1st example: muon production $e^+e^- \to \mu^+\mu^-$

μ : "heavy electron" ($m_\mu \simeq 106$ MeV $\simeq 200\, m_e$)

$\rightsquigarrow$ same Feynman rules

$\Rightarrow$ lowest order = "tree level" (i.e. no loops):



in $\xrightarrow[t]{}$ out

$\Rightarrow$ contributions to $i\mathcal{M}$:

1. Fermion chains (against arrow direction!)

   a) muons: $\bar{u}^r(k)(-ie\gamma^\sigma)v^{r'}(k')$

   b) electrons: $\bar{v}^{s'}(p')(-ie\gamma^\rho)u^s(p)$

2. photon propagator:

$$\frac{-ig_{\rho\sigma}}{(p+p')^2 \cancel{+i\varepsilon}} \equiv \frac{-ig_{\rho\sigma}}{s} \qquad \text{"s-channel"}$$

$$\Rightarrow i\mathcal{M} = \frac{ie^2}{s}\left(\bar{u}^r(k)\gamma^\sigma v^{r'}(k')\right)\left(\bar{v}^{s'}(p')\gamma_\sigma u^s(p)\right)$$

$$(\bar{u}\gamma^3 v)^* = (u^\dagger \gamma^0 \gamma^3 v)^\dagger$$
$$= v^\dagger \underbrace{\gamma^{3\dagger}\gamma^{0\dagger}}_{\gamma^0\gamma^3} u = \bar{v}\gamma^3 u$$

$$\Rightarrow |M|^2 = \frac{e^4}{s^2}\left[(\underline{\bar{u}(k)\gamma^\sigma v(k')})(\bar{v}(p')\gamma_\sigma u(p))\right]$$
$$\times\left[(\underline{\bar{v}(k')\gamma^3 u(k)})(\bar{u}(p)\gamma_3 v(p'))\right]$$

typically interested in <u>unpolarized</u> cross sections, i.e.

$$\boxed{|M|^2 \longrightarrow \frac{1}{(2s_A+1)(2s_B+1)} \sum_{s,s',r,r'} |M|^2 \equiv |\bar{M}|^2}$$

"sum over final, average over initial spin states"

only for massive particles!

$\leftrightarrow$ photon: 2 d.o.f.

$\rightarrow$ can use $\sum_s u^s(p)\,\bar{u}^s(p) = \not{p} + m$!

$$\Rightarrow \text{e.g. } \_\_\times\_\_ = \sum_{r,r'} \bar{u}^r_a(k)\,\gamma^\sigma_{ab}\,\underbrace{v^{r'}_b(k')\,\bar{v}^{r'}_c(k')}_{=(\not{k}'-m_\mu)_{bc}}\,\gamma^3_{cd}\,u^r_d(k)$$

$$= (\not{k}+m_\mu)_{da}$$

$$= \mathrm{Tr}[(\not{k} + m_\mu)\gamma^\sigma(\not{k}' - m_\mu)\gamma^\rho]$$

very generic expression that appears in calculating $|\bar{\mathcal{M}}|^2$

⇝ useful to collect properties of <u>traces</u> of γ matrices

- $\mathrm{Tr}[(\text{any odd \# of } \gamma\text{'s})] = 0$

  $\ulcorner = \mathrm{Tr}[\gamma^5\gamma^5(\ )] \quad | \quad \{\gamma^5, \gamma^\mu\} = 0$

  $= -\mathrm{Tr}[\gamma^5(\ )\gamma^5] \quad | \quad \mathrm{Tr}[A_1 \cdot A_2 \cdot \ldots A_{n-1} \cdot A_n] = \mathrm{Tr}[A_n \cdot A_1 \cdot A_2 \cdots A_{n-1}]$

  $= -\mathrm{Tr}[\underbrace{\gamma^5\gamma^5}_{1}(\ )] \quad \lrcorner$

- $\mathrm{Tr}[\mathbb{1}] = 4$
- $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

  $\ulcorner = \mathrm{Tr}[2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu]$

  $= 8g^{\mu\nu} - \underbrace{\mathrm{Tr}[\gamma^\nu\gamma^\mu]}_{\mathrm{Tr}[\gamma^\mu\gamma^\nu]} \quad \square \quad \lrcorner$

- $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = [\ldots]$

  $= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

- $\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\ldots] = \mathrm{Tr}[\ldots\gamma^{\mu_2}\gamma^{\mu_3}]$

$$\Rightarrow \bullet\ \mathrm{Tr}[(\not{k}+m_\mu)\gamma^\sigma(\not{k}'-m_\mu)\gamma^\rho]$$

$$= \underbrace{\mathrm{Tr}[\not{k}\gamma^\sigma\not{k}'\gamma^\rho]}_{k_\mu k'_\nu \mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho]} - m_\mu^2 \underbrace{\mathrm{Tr}[\gamma^\sigma\gamma^\rho]}_{4g^{\sigma\rho}}$$

$$= 4[k^\sigma k'^\rho - (k\cdot k')g^{\sigma\rho} + k^\rho k'^\sigma - m_\mu^2 g^{\sigma\rho}]$$

$$\bullet\ \mathrm{Tr}[(\not{p}'-m_e)\gamma_\sigma(\not{p}+m_e)\gamma_\rho] = (k^{(\prime)} \to p^{(\prime)},\ m_\mu \to m_e)$$

$$\Rightarrow |\bar{\mathcal{M}}|^2 = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4e^4}{s}[k^\rho k'^\sigma + k^\sigma k'^\rho - g^{\rho\sigma}(k\cdot k' - m_\mu)]$$

$$\times [p_\rho p'_\sigma + p_\sigma p'_\rho - g_{\rho\sigma}(p\cdot p' - m_e^2)]$$

$\downarrow$ $m_e \ll m_\mu$ [NB: $\alpha = \frac{e^2}{4\pi} \sim 10^{-2}$!]

$$\simeq \frac{8e^4}{s^2}[(k\cdot p)(k'\cdot p') + (k'\cdot p)(k\cdot p') + m_\mu^2\, p\cdot p']$$

now 2 options: a) choose a reference frame and evaluate contractions explicitly

$\rightsquigarrow$ angular dependence $\rightsquigarrow$ see P&S

b) keep everything Lorentz-invariant

@ b) $\bullet\ k\cdot p = -\frac{1}{2}[\underbrace{(k-p)^2}_{t} - \underset{k^2}{\underset{\parallel}{\cancel{m_e^2}}} - \underset{p^2}{\underset{\parallel}{m_\mu^2}}] = k'\cdot p'$

$p+p' = k+k'$

$\bullet\ k\cdot p' = [\sim t \to u] = -\frac{1}{2}[-s - t + \cancel{m_e^2} + m_\mu^2] = k'\cdot p$

• $p\cdot p' = \frac{1}{2}\big[\underbrace{(p+p')^2}_{s} - 2m_e^2\big]$

[• $k\cdot k' = \frac{1}{2}\left[s - 2m_\mu^2\right]$

$$\Rightarrow \ldots \quad \boxed{|\bar{\mathcal{M}}|^2 = \frac{4e^4}{s^2}\left[(t-m_\mu^2)^2 + \frac{s^2}{2} + st\right]}$$

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s}\,\frac{1}{\vec{p}_{cm}^{\,2}}\,|\bar{\mathcal{M}}|^2$$

cms: $\vec{p} = -\vec{p}\,'$ $\Rightarrow$ $E_p = E_{p'}$ $\quad m_e = m_e'$

$\Rightarrow s = (p+p')^2 = 4E_{cm}^2 = 4E_p^2$

$\vec{p}_{cms}^{\,2} = E_p^2 - m_e^2 = \frac{s}{4} - m_e^4$

$k_{cms}^2 = \frac{s}{4} - m_\mu^2$

$\Rightarrow t_{min/max} = \left(|\vec{p}|_{cm} \mp |\vec{k}|_{cm}\right)^2$

$\rightsquigarrow \quad \overset{!}{=} m_\mu^2 - \frac{s}{2}\left(1 \mp \sqrt{1-\frac{4m_\mu^2}{s}}\right)$

phase-space suppression

$$\Rightarrow \boxed{\sigma \simeq \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left(1+\frac{2m_\mu^2}{s}\right)} \quad \text{where } \alpha = \frac{e^2}{4\pi}$$

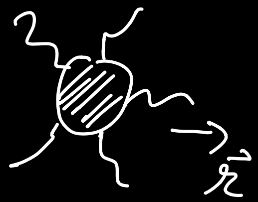
NB: • $\sigma \sim \frac{\alpha^2}{s}$ ✓

• $s \geq 4m_\mu^2 \iff E_{cm} \overset{!}{\geq} 2m_\mu$

for $\sigma > 0$ ✓

$\rightsquigarrow$ agreement with naive expectation! ✓

photon polarizations



$$\Rightarrow \quad i\mathcal{M} = i\,\mathcal{M}^\mu\,\varepsilon^*_\mu(k)$$

$$\int d^4x\, e^{ikx}\,\langle \text{final}\,|\,\underbrace{\bar\psi(x)\gamma^\mu\psi(x)}_{\equiv j^\mu}\,|\,\text{initial}\rangle$$

ε^*_μ

$$\Rightarrow k_\mu \mathcal{M}^\mu \propto \int d^4x\,(\partial_\mu e^{ikx})\,\langle \text{final}\,|\,j^\mu\,|\,\text{initial}\rangle$$

$$= -\int d^4x\, e^{ikx}\,\langle \text{final}\,|\,\underbrace{\partial_\mu j^\mu}_{=0}\,|\,\text{initial}\rangle$$

$=0$ (classical e.o.m) !

$$\Rightarrow \boxed{k_\mu \mathcal{M}^\mu = 0} \quad (*)$$

"Ward identity"

[general proof: QFT; consequence of gauge invariance / current conservation]

$$\Rightarrow \sum_{\text{photon polarizations}} |\mathcal{M}|^2 = \sum_\varepsilon \varepsilon^*_\mu(k)\,\varepsilon_\nu(k)\,\mathcal{M}^\mu(k)\,\mathcal{M}^\nu(k)^*$$

choose $k^\mu = (k,0,0,k)$ + transverse polarizations: $\varepsilon_1^\mu = (0,1,0,0)$, $\varepsilon_2^\mu = (0,0,1,0)$

$\varepsilon=\varepsilon_1$, $\varepsilon=\varepsilon_2$

$$= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2$$

$(*) \Rightarrow k\mathcal{M}^0 - k\mathcal{M}^3 = 0$

$$= -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2$$

$$= -g_{\mu\nu}\,\mathcal{M}^\mu \mathcal{M}^{\nu *}$$

i.e. $$\boxed{\sum_\varepsilon \varepsilon^*_\mu \varepsilon_\nu \longrightarrow -g_{\mu\nu}}$$

NB: not an equality!