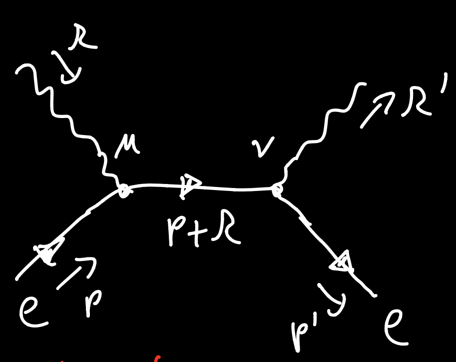
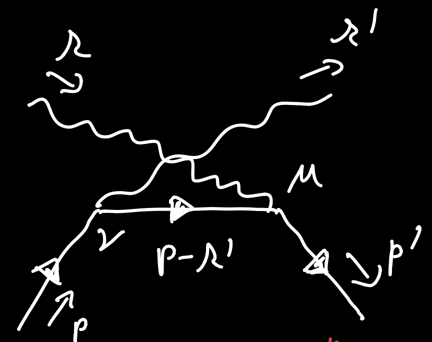
2^{nd} example: Compton scattering, $e^-\gamma \to e^-\gamma$



"s-channel"

$+$



"u-channel"

NB: fermion parts identical

$\Rightarrow$ no relative minus sign!

$$\Rightarrow i\mathcal{M} = (-ie)^2\, \bar{u}(p') \left\{ \frac{\gamma^\nu\, i(\not{p}+\not{k}+m)\,\gamma^\mu}{\underbrace{(p+k)^2}_{s}-m^2} + \frac{\gamma^\mu\, i(\not{p}-\not{k}'+m)\,\gamma^\nu}{\underbrace{(p-k')^2}_{u}-m^2} \right\} u(p)\, \varepsilon^*_\nu(k')\, \varepsilon_\mu(k)$$

use Dirac equation to simplify!

$$(\not{p}+m)\gamma^\mu u(p) = 2p^\mu u(p) + \gamma^\mu \underbrace{(-\not{p}+m)u(p)}_{=0}$$

$$\Rightarrow i\mathcal{M} = -ie^2\, \bar{u}(p') \left\{ \frac{\gamma^\nu \not{k} \gamma^\mu + 2\gamma^\nu p^\mu}{s-m^2} + \frac{-\gamma^\mu \not{k}' \gamma^\nu + 2\gamma^\mu p^\nu}{t-m^2} \right\} u(p)\, \varepsilon_\mu(k)\, \varepsilon^*_\nu(k')$$

$$\Rightarrow \overline{|\mathcal{M}|^2} = \frac{1}{\underset{e^-}{2}\cdot\underset{\gamma}{2}} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4} \sum_{\substack{\text{photon}\\ \text{spins}}} \underbrace{\varepsilon_\mu(k)\,\varepsilon^*_{\mu'}(k)}_{\to\, g_{\mu\mu'}}\ \underbrace{\varepsilon^*_\nu(k')\,\varepsilon_{\nu'}(k')}_{\to\, g_{\nu\nu'}}$$

$$\times \mathrm{Tr}\left[(\not p' + m)\left\{\frac{\gamma^\nu \not k \gamma^\mu + 2\gamma^\nu p^\mu}{s-m^2} + \frac{-\gamma^\mu \not k' \gamma^\nu + 2\gamma^\mu p^\nu}{u-m^2}\right\}\right.$$
$$\left. \times (\not p + m)\left\{\frac{\gamma^{\mu'} \not k \gamma^{\nu'} + 2\gamma^{\nu'} p^{\mu'}}{s-m^2} + \frac{-\gamma^{\nu'} \not k' \gamma^{\mu'} + 2\gamma^{\mu'} p^{\nu'}}{u-m^2}\right\}\right]$$

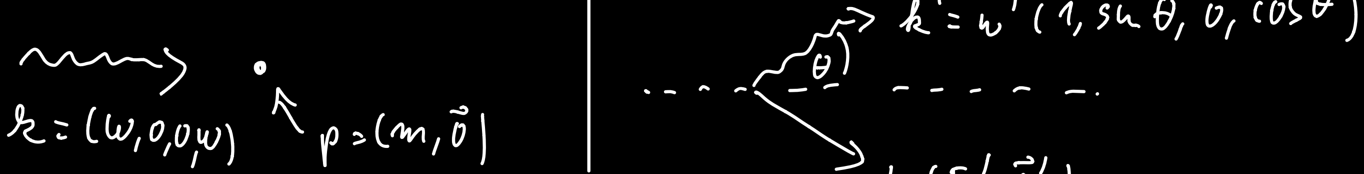
$(\bar\psi_1 \gamma^\mu \gamma^\nu \psi_2)^\dagger = \bar\psi_2 \gamma^\nu \gamma^\mu \psi_1$

- identify symmetries between 4 terms from expanding the sum:
  - cross terms $\propto \frac{1}{(s-m^2)}\frac{1}{(t-m^2)}$ identical
  - term $\propto \frac{1}{(u-m^2)^2}$ from $\frac{1}{(s-m^2)^2}$ term and $k \to -k'$
- $\not p \not p = p^2 = m^2$
- $\gamma^\mu \not p \gamma_\mu = -2\not p$
- ...

$$\rightsquigarrow \boxed{\overline{|\mathcal{M}|^2} = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + \left(1 + \frac{m^2}{p\cdot k} - \frac{m^2}{p\cdot k'}\right)^2 - 1\right]}$$

typically described in "lab" frame:

before collision	after collision



$$\Rightarrow p\cdot k' = m\omega'$$
$$p\cdot k = m\omega$$

$$\bullet\ \omega' = \omega'(\omega,\theta) \ : \ \cancel{m}^2 = p'^2 = (p+k-k')^2$$

$$= p^2 + 2p\cdot(k-k') + \underbrace{(k-k')^2}_{-2\,k\cdot k'}$$

$$= \cancel{m^2} + \cancel{2}\, m(\omega-\omega') - \cancel{2}\,\omega\omega'(1-\cos\theta)$$

$$\Rightarrow \omega'(m+\omega(1-\cos\theta)) = m\omega$$

$$\Leftrightarrow \boxed{\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}} \quad (*)$$

$$\bullet\ \int d\Pi_2 = \int \frac{d^3k'}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\,\frac{1}{2\omega'\,2E'}\,(2\pi)^4\,\delta^{(4)}(k+p-k'-p')$$

$$= \int \frac{\omega'^2\,d\Omega\,d\omega'}{(2\pi)^3}\,\frac{1}{4\omega' E'}\,2\pi\,\delta\Big(\omega+\underset{\substack{\|\\ m}}{E}-\omega'-E'\Big) \quad E' = \sqrt{m^2+(\vec{k}-\vec{k}'+\underset{\substack{\|\\ 0}}{\vec{p}})^2}$$

$$= \sqrt{m^2+\omega^2+\omega'^2-2\omega\omega'\cos\theta}$$

$$\left|\ \int f(x)\,\delta(g(x))\,dx = \frac{1}{|g'(x_0)|}\,f(x_0)\right.$$

$$= \int \frac{d\cos\theta}{2\pi}\,\frac{\omega'}{4E'}\left|+1+\frac{\cancel{2}\omega'-\cancel{2}\omega\cos\theta}{\cancel{2}E'}\right|^{-1}$$

$$= \int \frac{d\cos\theta}{8\pi}\,\frac{\omega'}{\underbrace{E'+\omega'}_{=E+\omega=m+\omega}-\omega\cos\theta} = \int\frac{d\cos\theta}{8\pi}\,\frac{\omega'}{m+\omega(1-\cos\theta)}$$

$$\overset{(*)}{=} \int \frac{d\cos\theta}{8\pi}\,\frac{(\omega')^2}{m\omega}$$

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{1}{2\underset{= m}{E}\, 2\omega\, \underbrace{|v_A - v_B|}_{1}} \frac{1}{8\pi} \frac{|\omega'|^2}{m\omega} \overline{|M|}^2$$

$$= \frac{e^4}{16\pi m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + \left(1 + \frac{m}{\omega} - \frac{m}{\omega'}\right)^2 - 1\right]$$

$$\left| \frac{m}{\omega} - \frac{m}{\omega'} = \frac{m}{\omega}\left(1 - \frac{\omega}{\omega'}\right)\right.$$

$$\overset{(*)}{=} \frac{m}{\omega}\left[1 - 1 - \frac{\omega}{m}(1 - \cos\theta)\right]$$

$$= -(1 - \cos\theta)$$

$$\Rightarrow \boxed{\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]}}$$

"Klein-Nishima" formula

$\omega \ll m$

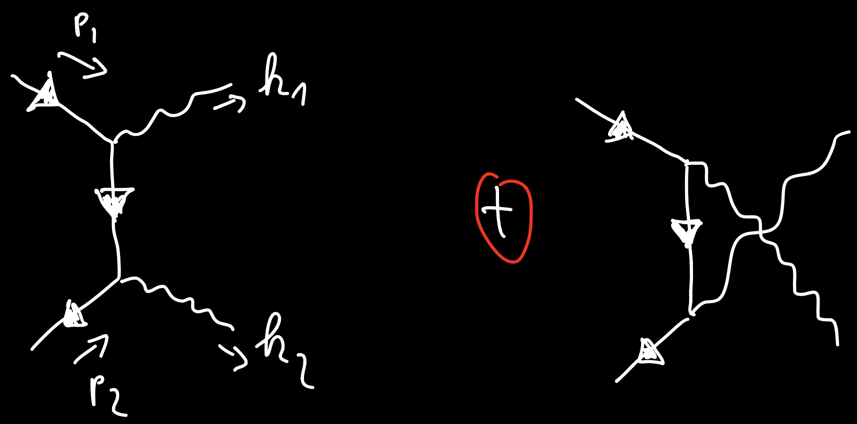
$\Rightarrow \omega = \omega'$

$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta)}$$

Thompson cross section (classical EM!)

✓

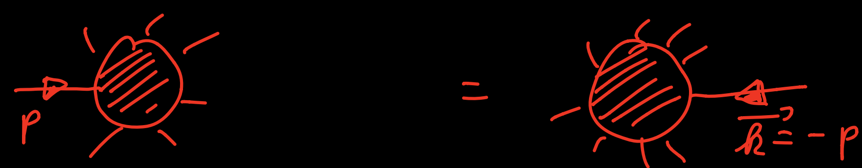
3rd example – pair annihilation into photons : $e^+e^- \to \gamma\gamma$



$\rightsquigarrow$ this is related to Compton scattering by "crossing symmetry"

antiparticle of ϕ

general : $\mathcal{M}(\phi(p) + \ldots \to \ldots) = \mathcal{M}(\ldots \to \ldots + \bar{\phi}(-p))$



($p + \sum$ all other momenta $= 0$) ($\sum$ all other momenta $- k = 0$)

only difference in external legs :

$$\sum u(p)\,\bar{u}(p) = \not{p} + m = -(\not{k} - m)$$

$$= - \sum v(k)\,\bar{v}(k)$$

for each crossed fermion !

here : (compared to notation from Compton scattering case)

$$p \to p_1$$
$$p' \to -p_2$$
$$k' \to k_2$$
$$k \to -k_1$$
$$+\ \overline{|\mathcal{M}|}^2 \to -\overline{|\mathcal{M}|}^2$$

$$\Rightarrow \overline{|\mathcal{M}|}^2 = -2e^4\left[-\frac{p_1\cdot k_2}{p_1\cdot k_1} - \frac{p_1\cdot k_1}{p_1\cdot k_2} + \left(1 - \frac{m^2}{p_1\cdot k_1} - \frac{m^2}{p_1\cdot k_2}\right)^2 - 1\right]$$

high-energy limit : $s \gg 4m^2$

$\Rightarrow$ CMS : $p_1 = (E, \vec{p})$; $p_2 = (E, -\vec{p})$; $E = |\vec{p}|$

$k_1 = (\omega, \vec{k})$; $k_2 = (\omega, -\vec{k})$; $\omega = |\vec{k}|$

$\Rightarrow p_1\cdot k_2 = E\omega\,(1+\cos\theta)$

$p_1\cdot k_1 = E\omega\,(1-\cos\theta)$

$$\Rightarrow \overline{|\mathcal{M}|}^2 = 2e^4\left[\underbrace{\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}}_{\frac{2+2\cos^2\theta}{\sin^2\theta}} + 0\right]$$

$m \ll E$!

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{1}{\underbrace{2E_A\,2E_B}_{=E_{cm}^2=s}\underbrace{|v_A - v_B|}_{2}}\ \underbrace{\frac{|\vec{k}_1|}{8\pi E_{cm,2}}}\ 4e^4\,\frac{1+\cos^2\theta}{\sin^2\theta}$$

$d\Pi_2/d\cos\theta$ in CMS

$$\Rightarrow \boxed{\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \frac{1+\cos^2\theta}{\sin^2\theta}}$$