8. Functional methods

alternative way to calculate transition amplitude for ϕ_a at $x^0 = 0$ to ϕ_b at $x^0 = T$:

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int \mathcal{D}\phi \, \exp\Big[i \underbrace{\int_0^T d^4x \, \mathcal{L}[\phi]}_{=S} \Big]$$

"path integral"

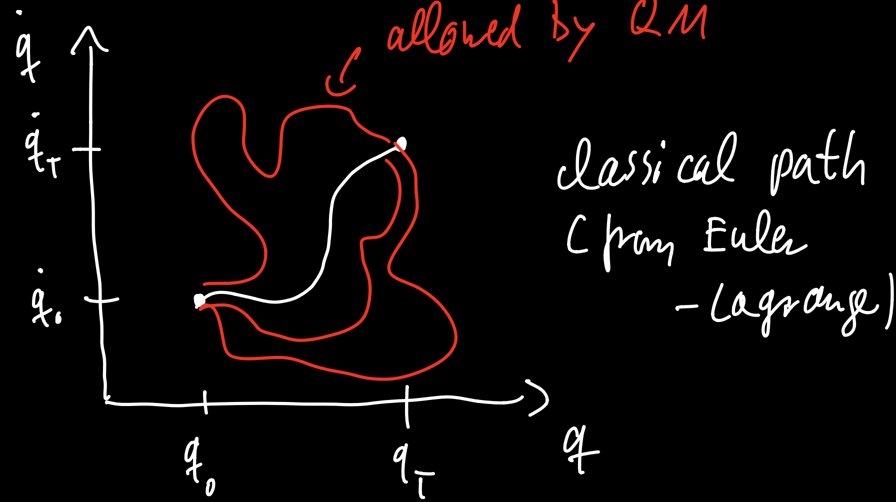
= sum over all possible $\phi(x)$

with $\phi(0,\vec{x}) = \phi_a(\vec{x})$

and $\phi(T,\vec{x}) = \phi_b(\vec{x})$

$$\rightsquigarrow \boxed{\mathcal{D}\phi = \prod_i d\underbrace{\phi(x_i)}_{\hat{=}\, q_i}}$$

$\hat{=}$ classical mechanics:



$\rightsquigarrow$ correlation functions: (claim)

$$\boxed{\langle \Omega | T\{\hat{\phi}_H(x_1)\hat{\phi}(x_2)\ldots\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\ldots \exp[i\int_{-T}^{T} d^4x\,\mathcal{L}]}{\int \mathcal{D}\phi \exp[i\int_{-T}^{T} d^4x\,\mathcal{L}]}}$$

i.e. "$\langle 0|T\{\ldots\}|0\rangle$ $\longrightarrow \int \mathcal{D}\phi$"

$-H_I \longrightarrow S = \int d^4x\,\mathcal{L}$

compared to previous "master formula"

application: Feynman rules for ϕ^4 theory

a) first $\lambda = 0 \Rightarrow \mathcal{L} = \mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$ $\quad (\partial_\mu\phi)^2 = \partial_\mu\phi\,\partial^\mu\phi$

$\Rightarrow$ integrals are of Gaußian type!

i) use $\left(\prod_{k=1}^{N}\int d\xi_k\right)\exp[-\xi_i B_{ij}\xi_j]$ | diagonalize (possible for $B_{ij}=B_{ji}$)

$\xi_i = O_{ij}x_j$; $O^T = O^{-1}$ (*)

(*) $\Rightarrow$ Jacobian $=1$

$$= \left(\prod_k \int dx_k\right)\exp[-x_i \underbrace{(O^{-1}BO)_{ij}}_{\delta_{ij}b_i} x_j]$$

$\delta_{ij}b_i$; b_i: eigenvalues of B!

no sum over i

$$= \left(\prod_k \int dx_k\right)\left(\prod_i \exp[-b_i x_i^2]\right)$$

$$= \prod_i \Big(\underbrace{\int dx_i \, \exp[-b_i x_i^2]}_{=\sqrt{\pi/b_i}}\Big)$$

$$= \text{const.} \times [\det B]^{-1/2}$$

in analogy: $\quad i S_0 = -\frac{i}{2}\int d^4x \, \phi(\partial^2 + m^2)\phi$ [+ surface term]

partial integration

$$= -i \underbrace{\int d^4x}_{\text{"}i\text{"}} \underbrace{\int d^4y}_{\text{"}j\text{"}} \underbrace{\phi(y)}_{\text{"}\xi_i\text{"}} \underbrace{\tfrac{1}{2}\delta^{(4)}(x-y)(\partial^2+m^2)}_{\text{"}=B_{ij}\text{"}} \underbrace{\phi(x)}_{\text{"}\xi_j\text{"}}$$

$$\Rightarrow \boxed{\int \mathcal{D}\phi \, e^{iS_0} = \text{const.} \times [\det(m^2+\partial^2)]^{-1/2}}$$

"functional determinant"

[explicitly expressable in Fourier space, c.f. (9.23), ⇝ "undefined parts" cancel ...]

ii) now consider

$$\left(\prod_k \int d\xi_k\right) \xi_\ell \, \xi_m \, \exp[-\xi_i B_{ij} \xi_j]$$

$$= \left(\prod_k \int d x_k\right) (O_{\ell\ell'} x_{\ell'}) (O_{mm'} x_{m'}) \exp[-\sum_i b_i x_i^2]$$

- $k \neq \ell', m' \Rightarrow \int_{-\infty}^{\infty} d x_k \exp[-\sum_i b_i x_i^2]$

$$= \sqrt{\frac{\pi}{b_k}} \exp[-\sum_{i \neq k} b_i x_i^2]$$

- $k = \ell' \neq m'$ or $k = m' \neq \ell'$

$\Rightarrow \int dx_k = 0$ (antisymmetric!)

- $k = \ell' = m'$:

$$O_{\ell k} O_{mk} \underbrace{\int dx_k \, x_k^2 \exp[-\sum_i b_i x_i^2]}_{= \frac{\sqrt{\pi}}{2} b_k^{-3/2} \exp[-\sum_{i \neq k} b_i x_i^2]}$$

$$= \text{const} \times [\det B]^{-1/2} \times \underbrace{\sum_k O_{\ell k} O_{mk} b_k^{-1}}_{(O \underbrace{(b^{-1} \mathbb{1})}_{= O^T B^{-1} O} O^T)_{\ell m}}$$

$$= \text{const} \times [\det B]^{-1/2} \times (B^{-1})_{\ell m}$$

$$\Rightarrow \boxed{\int \mathcal{D}\phi \, \phi(x)\phi(y)\, e^{iS_0} = \text{const} \times \left[\det(m^2+\partial^2)\right]^{-1/2} \times G(x-y)}$$

where $G(x-y) = \text{``}(\partial^2+m^2)^{-1}\text{''}$

$= $ Green's function of KG operator:

$$(\partial_x^2+m^2)\, G(x-y) = -i\,\delta^{(4)}(x-y)$$

NB: require the Gaussian integration to be well defined

(i.e. $\int dx\, x^{2n} \exp[-bx^2]$, not $\int dx\, x^{2n} \exp[-ibx^2]$)

$\Rightarrow$ need to replace $-k^2+m^2 \to -k^2+m^2-i\epsilon$

$\Rightarrow G(x-y) = D_F(x-y)$

$$\Rightarrow \boxed{\langle \cancel{\Omega}^{\,0} | T\phi(x)\phi(y)|\cancel{\Omega}^{\,0}\rangle = D_F(x_1-x_2) \text{ also with new formalism } \checkmark \equiv \phi(x)\phi(y)}$$

$(\lambda=0)$

four point functions:

$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = $ sum of all full contractions

[everything else gives anti-symmetric integrands as $\mathcal{D}\phi = 0$]

$$= D_F(x_1-x_2)\,D_F(x_3-x_4)$$
$$+ D_F(x_1-x_3)\,D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)\,D_F(x_2-x_3)$$

= same as from Wick's theorem ✓

b) $(\lambda \neq 0)$

expand to next order in λ:

$$\exp[i\int d^4x\,\mathcal{L}] = \exp[i\int d^4x(\mathcal{L}_0 + \mathcal{L}_{int})]$$
$$= \exp[i\int d^4x\,\mathcal{L}_0]\left(1 - i\int d^4x\,\frac{\lambda}{4!}\phi(x)^4 + \dots\right)$$

=> as before, everything can be expressed in terms of free-field correlation functions!

⇝ same combinatorics, exponentiation of vacuum bubble diagrams etc.

e.g. vertex $= -i\lambda$

⇒ general procedure (valid for all QFTs!)

1. analyze quadratic terms in $\mathcal{L}$ ($\rightsquigarrow \mathcal{L}_0$)

   $\rightsquigarrow$ derive propagators

   (= inverse of these terms)

2. vertices can be directly identified from $\mathcal{L}$ as coefficients in front of cubic and higher order terms

   (times a factor of "i")

math excursion:

calculus of functionals

$$\text{function } f \text{ [physics]}: \begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\text{map}} & \mathbb{R}^m \\ \cup & & \\ \Omega & & \Downarrow \\ \Downarrow & & \\ x & & f(x) \end{array}$$

$$\text{functional } F: \begin{array}{ccc} \mathcal{D}(\Omega) & \longrightarrow & \mathbb{R}, \mathbb{C} \\ \Downarrow & & \\ f & & F[f] \end{array}$$

$\mathbb{R}$

$\mathcal{D}(\Omega)$: space of all test functions over Ω

distribution $\equiv$ continuous, linear functional

$\Rightarrow$ [regular] distribution $\equiv$ $\exists$ locally integrable function ϕ:

$$F[f] = \int_\Omega \phi(x)\, f(x)\, dx \quad \forall f \in \mathcal{D}(\Omega)$$

$$\text{"}= F[f](x)\text{"} = \text{"}F[f(x)]\text{"}$$

symbolic (!) example: Dirac delta "function" $\delta(x)$

$$\delta[f] \equiv f(0) \text{ "="} \int \delta(x)\, f(x)\, dx$$

but δ is not really regular:

$$\text{e.g. } \delta_a(x) \equiv \frac{1}{\sqrt{2\pi}a}\, e^{-\frac{x^2}{2a^2}}$$

$$\Rightarrow \delta[f] \equiv \lim_{a\to 0} \int \delta_a(x)\, f(x)\, dx = f(0)$$

$\neq \int \underbrace{\lim_{a \to 0} \delta_a(x)}_{\begin{cases} 0 & \text{for } x \neq 0 \\ [\infty & \text{for } x = 0] \end{cases}} f(x)\, dx = 0$

(first) variation of a functional:

$$\delta F[\underset{\uparrow \atop \mathcal{D}(x)}{f}, \underset{\uparrow \atop \mathcal{D}(x)}{\varphi}] \equiv \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} F[f + \varepsilon\varphi]\Big|_{\varepsilon=0} \equiv \lim_{\varepsilon \to 0} (F[f+\varepsilon\varphi] - F[f])$$

functional derivative (= regular distribution!) : $\dfrac{\delta}{\delta g(x)} F[g]$:

$$\int \underbrace{\frac{\delta}{\delta g(x)} F[g(y)]}_{\phi(y)} f(y)\, dy$$

$$\equiv \frac{d}{d\varepsilon} F[g + \varepsilon f](x)\Big|_{\varepsilon=0} \quad \forall f \in \mathcal{D}(\Omega)$$

calculus : $\bullet \ \dfrac{\delta}{\delta f(x)} f(y) \equiv \delta^{(4)}(x-y)$ "basic axiom"

$$\Leftrightarrow \frac{\delta}{\delta f(x)} \int d^4y\, f(y)\, \varphi(y) \equiv \varphi(x) \quad (*)$$

$\bullet \ \dfrac{\delta}{\delta f(x)}\, g\left(\int d^4y\, f(y)\, \varphi(y)\right) = \varphi(x) \cdot g'\left(\int d^4y\, f(y) g(y)\right)$

$\bullet \ \dfrac{\delta}{\delta f(x)} \int d^4y\, (\partial_\mu f(y))\, V^\mu(y) = -\partial_\mu V^\mu(x)$

$$\bullet\ \frac{\delta}{\delta f(x)}\Big(g(y)\,h(y)\Big) = g(y)\,\frac{\delta h(y)}{\delta f(x)} + h(y)\,\frac{\delta g(y)}{\delta f(x)}$$

[confused? try to think of N-dim analogy!

e.g. (*) : discretize ⇝ $\bullet \int d^4y \to \sum_i$

$\bullet\ f(y) \to x_i$

$$\Rightarrow \frac{\delta}{\delta f(x)} \to \frac{d}{dx_i}$$

$$\Rightarrow (*)\ \frac{\delta}{\delta f(x)} \int d^4y\, f(y)\, \varphi(y)$$

$$\to \frac{d}{dx_i}\left(x_j\, \varphi_j \right) = \varphi_i$$

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