The generating functional

$$Z[J] \equiv \int \mathcal{D}\phi \, \exp\Big[i \int d^4x \, \big[\mathcal{L} + \overbrace{\underset{\substack{\uparrow \\ \text{"source field"}}}{J(x)} \, \phi(x)}^{\text{"source term"}} \big] \Big]$$

recall : $\dfrac{\delta}{\delta J(x)} \int d^4y \, J(y) \phi(y) = \phi(x)$

$\Rightarrow$ very simple to calculate correlation functions:

$$\boxed{\begin{aligned} \langle \Omega | T\{ \phi(x_1) \phi(x_2) \dots \} | \Omega \rangle &= \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \, \phi(x_1) \phi(x_2) .. \exp[i \int_{-T}^{T} d^4x \, \mathcal{L}]}{\int \mathcal{D}\phi \, \exp[i \int_{-T}^{T} d^4x \, \mathcal{L}]} \\ &= \frac{1}{\underset{\substack{||| \\ Z|_{J=0}}}{Z_0}} \left(-i \frac{\delta}{\delta J(x_1)} \right) \left(-i \frac{\delta}{\delta J(x_2)} \right) \dots Z[J] \Big|_{J=0} \end{aligned}}$$

application : free-field theory

- $\lambda = 0 \leadsto |\Omega\rangle = |0\rangle$

$$\leadsto \int d^4x \, [\mathcal{L} + J\phi] = \int d^4x \, [+\tfrac{1}{2} \phi(-\partial^2 - m^2 + \underset{\uparrow}{i\epsilon})\phi + J\phi]$$

necessary for convergence of $\int \mathcal{D}\phi$

⌈ analogy : want to compute $\int dx \, \exp[-Ax^2 + Bx]$ ⌋

$\rightarrow$ standard method: "complete the square"

(remove terms linear in ϕ !)

symbolically : $\phi' \equiv \phi + \varphi$

$$\Rightarrow \mathcal{L} + J\phi = \frac{1}{2}(\phi' - \varphi)\underbrace{(-\partial^2 - m^2 \boxed{+i\varepsilon})}_{\text{"B"}}(\phi' - \varphi) + J(\phi' - \varphi)$$

$$= \frac{1}{2}\phi' B \phi' + \frac{1}{2}\varphi B \varphi$$

$$- \frac{1}{2}\underline{\varphi B}\,\phi' - \frac{1}{2}\phi' \underbrace{B\varphi}_{J} + \underline{J\phi'} - J\varphi$$

choose $\varphi = B^{-1}J$

$$= \frac{1}{2}\phi' B \phi' - \frac{1}{2}\varphi B^{-1} \varphi$$

rigorously : $\phi'(x) \equiv \phi(x) - i\int d^4y\, D_F(x-y) J(y)$

$\downarrow$ exercise : use the fact that D_F is a Green's function of KG operator

$$Z[J] = \underbrace{\int \mathcal{D}\phi}_{=\int \mathcal{D}\phi'} \exp\Big[\frac{i}{2}\int d^4x \Big\{\phi'(-\partial^2 - m^2 + i\varepsilon)\phi' - \frac{i}{2}\int d^4y\, J(x)(-iD_F(x-y))J(y)\Big\}\Big]$$

$$\boxed{\begin{aligned} Z[J] &= \underbrace{\int \mathcal{D}\phi' \exp\Big[i\int d^4x\, \frac{1}{2}\phi'(-\partial^2 - m^2 + i\varepsilon)\phi'\Big]}_{= Z_0} \\ &\times \underbrace{\exp\Big[-\frac{1}{2}\int d^4x \int d^4y\, J(x) D_F(x-y) J(y)\Big]}_{\text{independent of } \phi'} \end{aligned}}$$

$$\Rightarrow \langle 0 | T\{\phi(x_1)\,\phi(x_2)\ldots\} | 0 \rangle$$

$$= \left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\ldots \exp\left[-\frac{1}{2}\int d^4x \int d^4y \, J(x) D_F(x-y) J(y)\right]\Bigg|_{J=0}$$

$\Rightarrow$ exactly the same expressions as from Wick's theorem!

⌜analogy in 1D :

$$\left(\frac{\partial}{\partial x}\right)^n \exp[-Ax^2]\Big|_{x=0}$$

⌟

Functional quantization of spinor fields

Grassmann numbers

= anti-commuting numbers

$$\theta\eta = -\eta\theta$$

$\Rightarrow$ • $\theta^2 = 0$

• $\int_{-\infty}^{\infty} d\theta\, f(\theta) = \int d\theta\,(\overbrace{A + B}^{\text{c-numbers}}\theta) \equiv B$

↑ only choice that ensures invariance under $\theta \to \theta + \eta$

• $(\theta\eta)^* \equiv \eta^*\theta^* = -\theta^*\eta^*$

• complex integration : $\int d\theta^* d\theta\,(\theta\theta^*) \equiv 1 = -\int d\theta^* d\theta\,(\theta^*\theta)$

• Gaussian integrals :

a) $\int d\theta^* d\theta\, e^{-\theta^* b\theta} = \int d\theta^* d\theta\,(1 - \theta^* b\theta)$

$= \int d\theta^* d\theta\,(1 + \theta b\theta^*) = b$

b) $\int d\theta^* d\theta\, \theta^*\theta\, e^{-\theta^* b\theta} = 1 = \frac{b}{b}$

$\Rightarrow$ extra factor of $\theta^*\theta$ introduces a factor b^{-1} just like for standard Gaussian integrals!

$\rightsquigarrow$ [...] • $\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_m^* B_{mn}\theta_n} = \prod_i b_i = \det B$

• $\left(\prod_i \int d\theta_i^* d\theta_i\right) \theta_k \theta_l^*\, e^{-\theta_m^* B_{mn}\theta_n} = \det B\, (B^{-1})_{kl}$

→ describe <u>classical</u> spinors as Grassmann fields:

$$\boxed{\psi(x) = \sum_a \underset{\substack{\uparrow\\ \text{Grassmann}\\ \text{number}}}{\psi_a}\; \underset{\nwarrow}{\phi_a}(x) \qquad \text{c-functions = basis of 4-component spinors}}$$

generating functional:

$$\overset{[+\,\mathcal{L}_{int}]}{} \qquad Z[\bar\eta,\eta] \underset{\substack{\uparrow\\ \mathcal{L}=\mathcal{L}_0}}{\overset{(*)}{=}} \int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, \exp\Big[i\int d^4x \big\{\underbrace{\bar\psi(i\partial\!\!\!/-m)\psi}_{\mathcal{L}_0} + \bar\eta\psi + \bar\psi\eta\big\}\Big]$$

$\big\downarrow$ complete the square [exercise!]

$$= Z[\bar\eta,\eta=0] \times \exp\Big[-\int d^4x\, d^4y\, \bar\eta(x)\, S_F(x-y)\, \eta(y)\Big]$$

$$\Rightarrow \langle \overset{\Omega}{0}|T\{\psi(x_1)\bar\psi(x_2)\dots\}|\overset{\Omega}{0}\rangle = \lim_{\dots} \frac{\int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, \psi(x_1)\bar\psi(x_2)\dots \exp[i\int d^4x\, \mathcal{L}_0]}{\int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, \exp[i\int d^4x\, \mathcal{L}_0]}$$

$$\overset{(*)}{=} Z[\eta,\bar\eta=0]^{-1}\left(-i\frac{\delta}{\delta\bar\eta(x_1)}\right)\left(+i\frac{\delta}{\delta\eta(x_2)}\right)\dots Z[\eta,\bar\eta]\Big|_{\eta,\bar\eta=0}$$

where $\frac{\delta}{\delta\eta}\theta\eta = -\frac{\delta}{\delta\eta}\eta\theta = -\theta$

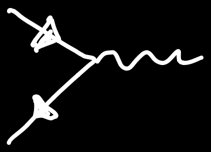
QED - again

$$\mathcal{L}_{QED} = \underbrace{\bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_0} - e\bar{\psi}\gamma^\mu\psi A_\mu$$

$$\Rightarrow \exp[i\int d^4x\,\mathcal{L}] = \underbrace{\exp[i\int d^4x\,\mathcal{L}_0]}\left(1 - ie\int d^4z\,\bar{\psi}(z)\gamma^\mu\psi(z)A_\mu(z) + \ldots\right)$$

→ fermion and photon propagators

↓ vertex rule can directly be read off from $\mathcal{L}_{int}$

$\Rightarrow$  $= -ie\gamma^\mu\,[\int d^4z]$

$$\Big[\, Z[\bar{\eta},\eta,J^\mu] = \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\,\exp[i\int d^4x\{\mathcal{L}_0 - e\bar{\psi}\gamma^\mu\psi A_\mu + \bar{\eta}\psi + \bar{\psi}\eta + A_\mu J^\mu\}]$$

$$= \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\left(1 - ie\int d^4z(\bar{\psi}\gamma^\sigma\psi A_\sigma)_z + \ldots\right)\exp[i\int d^4x\{\mathcal{L}_0 + \bar{\eta}\psi + \bar{\psi}\eta + A_\mu J^\mu\}]$$

$$\Big|\ \int dx\,x\,\exp[f(x) + yx]$$
$$= \partial_y\int dx\,\exp[f(x) + yx]$$

$$= \left(1 - ie\int d^4z\left(+i\frac{\delta}{\delta\eta(z)}\right)\gamma^\sigma\left(-i\frac{\delta}{\delta\bar{\eta}(z)}\right)\left(-i\frac{\delta}{\delta J^\sigma(z)}\right)\right) \times$$

$$\times\underbrace{\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\,\exp[i\int d^4x\,\mathcal{L}_0 + \bar{\eta}\psi + \bar{\psi}\eta + A_\mu J^\mu\}]}$$

[complete the square]

$$= \int D\bar{\psi}\, D\psi\, DA \, \exp[i \int d^4x \, \mathcal{L}_0] \times \quad \equiv Z_0^0$$

$$\left.\begin{aligned} &\times \exp[-\int d^4x\, d^4y \; \bar{\eta}(x) S_F(x-y) \eta(y)] \\ &\times \exp[-\tfrac{1}{2} \int d^4x\, d^4y \; J^\mu(x) \underset{\uparrow \text{ photon propagator}}{D^F_{\mu\nu}} (x-y) J^\nu(z)] \end{aligned}\right\} \text{independent of } \bar{\psi}, \psi, A$$

$$\Rightarrow \langle \Omega | T\{ \bar{\psi}(x_1) \psi(x_2) A_\rho(x_3) \} | \Omega \rangle$$

$$= Z_0^{-1} \left(+i \frac{\delta}{\delta \eta(x_1)} \right) \left(-i \frac{\delta}{\delta \bar{\eta}(x_2)} \right) \left(-i \frac{\delta}{\delta J^\rho(x_3)} \right) Z[\bar{\eta}, \eta, J^\mu] \Big|_{\bar{\eta}, \eta, J^\mu = 0}$$

$$= \frac{Z_0^0}{Z_0} (-ie) \int d^4z \left(+i \frac{\delta}{\delta \eta(x_1)} \right) \left(-i \frac{\delta}{\delta \bar{\eta}(x_2)} \right) \left(-i \frac{\delta}{\delta J^\rho(x_3)} \right) \left(+i \frac{\delta}{\delta \eta(z)} \right) \gamma^\sigma \left(-i \frac{\delta}{\delta \bar{\eta}(z)} \right) \left(-i \frac{\delta}{\delta J^\sigma(z)} \right)$$

$$\times \exp[-\int d^4x\, d^4y \; \bar{\eta}(x) S_F(x-y) \eta(y)]$$

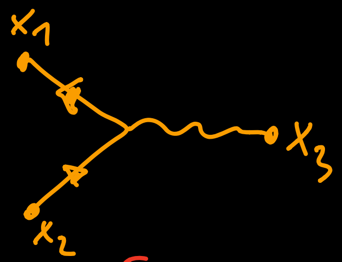
$$\times \exp[-\tfrac{1}{2} \int d^4x\, d^4y \; J^\mu(x) D^F_{\mu\nu} (x-y) J^\mu(z)] \Big|_{J, \eta, \bar{\eta} = 0}$$

$$= \frac{Z_0^0}{Z_0} (-ie) \int d^4z \, \frac{\delta}{\delta J^\rho(x_3)} \underbrace{\frac{\delta}{\delta J^\sigma(z)} \exp[-\tfrac{1}{2} \int d^4x\, d^4y \; J^\nu(x) D^F_{\nu\mu}(x-y) J^\mu(y)] \Big|_{J=0}}_{\underbrace{\left(-\int d^4y' D^F_{\sigma\mu}(z-y') J^\mu(y') \right) \exp[-\frac{1}{2} \int d^4x\, d^4y \; J^\nu(x) D^F_{\mu\nu}(x-y) J^\mu(y)] \Big|_{J=0}}_{-D^F_{\sigma\rho}(z-x_3) \exp[0]}}$$

$$\times \frac{\delta}{\delta\eta(x_1)} \frac{\delta}{\delta\bar\eta(x_2)} \gamma^\sigma \frac{\delta}{\delta\eta(z)} \underbrace{\frac{\delta}{\delta\bar\eta(z)} \exp\left[-\int d^4x\, d^4y\, \bar\eta(x) S_F(x-y)\eta(y)\right]}_{\left(-\int d^4y' S_F(z-y)\eta(y')\right)\exp\left[-\int d^4x\, d^4y\, \bar\eta(x) S_F(x-y)\eta(y)\right]}\Bigg|_{\eta,\bar\eta=0}$$

$$\to \underbrace{S_F(x_1-x_2)\gamma^\sigma S_F(z-z)}_{} - S_F(x_1-z)\gamma^\sigma S_F(z-x_2)$$

⇝ disconnected part, does not contribute to vertex rule!



$$\Rightarrow \langle\Omega|T\{\bar\psi(x_1)\psi(x_2)A_\rho(x_3)\}|\Omega\rangle$$

$$= \underbrace{\frac{Z_0^0}{Z_0}}_{=1+\mathcal{O}(e)} \underline{(-ie)}\int \underline{d^4z}\, D^F_{\sigma\rho}(z-x_3)\, S_F(x_1-z)\underline{\gamma^\sigma} S_F(z-x_2)$$

= vertex rule as claimed!

[higher orders:
⇝ disconnected diagrams
(⇝ exponentiation of vacuum bubbles!)]