Quantization of the electromagnetic field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \Big|\ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)F^{\mu\nu}$$

$$= +\frac{1}{2}A_\nu \partial_\mu F^{\mu\nu} \left[-\frac{1}{2}\partial_\mu (A_\nu F^{\mu\nu})\right]_{\text{surface}}$$

$$= \frac{1}{2}A_\nu(\partial^2 g^{\mu\nu} - \partial^\nu\partial^\mu)A_\mu$$

$$\Rightarrow S = \int d^4x\, \mathcal{L} = \frac{1}{2}\int d^4x\, A_\nu(x)\,(\partial^2 g^{\nu\mu} - \partial^\nu\partial^\mu)A_\mu(x)$$

$$\Big|\ A_\mu(x) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{A}_\mu(k)$$

$$= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4} A_\nu(k)(-k^2 g^{\nu\mu} + k^\nu k^\mu)\underbrace{A_\mu(-k)}_{= A_\mu^*(k)}$$

<u>Problem</u> : a) matrix $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ is singular, i.e. cannot be inverted!

related: b) $(-k^2 g^{\mu\nu} + k^\mu k^\nu)A_\nu = 0$ if $A_\nu(k) = f(k)\cdot k_\nu \quad \forall f(k)$

$\Rightarrow \exp[iS] = 1$ for infinitely many $A_\nu(x)$

$\Rightarrow \int \mathcal{D}A\, e^{iS}$ diverges!

solution by Faddeev & Popov:

1. recall gauge invariance

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ / Maxwell's equations invariant

under $\boxed{A_\mu(x) \longrightarrow A^\alpha_\mu(x) \equiv A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)}$ $\forall \alpha(x)$

$\Rightarrow$ gauge can be fixed by a gauge fixing condition

$$G(A^\alpha) \overset{!}{=} 0$$

e.g. Lorentz gauge: $G(A^\alpha) = \partial_\mu (A^\alpha)^\mu$

2. "trick": introduce unity in functional integral:

$$1 = \int \mathcal{D}\alpha(x)\, \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

[∞-dimensional version of $1 = \int da\, \delta(g(a)) \left|\frac{\partial g}{\partial a}\right|$]

$$\Rightarrow \int \mathcal{D}A\, e^{iS_0[A]} = \int \mathcal{D}\alpha\, \mathcal{D}A\, e^{iS_0[A]}\, \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

standard choice of parameterization:

$G(A^\alpha) = \partial^\mu A^\alpha_\mu - \omega(x)$

arbitrary function

$$\Rightarrow \frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{e}\partial^2$$: independent of α!

$$= \det\left(\tfrac{1}{e}\partial^2\right) \int \mathcal{D}\alpha \, \underbrace{\mathcal{D}A}_{\mathcal{D}A^\alpha} \, e^{i\underbrace{S[A]}_{=S[A^\alpha]}} \, \delta(G(A^\alpha))$$

$= S[A^\alpha]$ (gauge invariance !)

$\mathcal{D}A^\alpha$

("constant" shift in A does not induce a Jacobian)

- valid for all ω => also for any linear combinations

⇝ use $1 = N_\xi \int \mathcal{D}\omega \, \exp\left[-i\int d^4x \, \frac{\omega^2(x)}{2\xi}\right]$ for correct normalization (ξ = const.)

$$\Rightarrow \int \mathcal{D}A \, e^{iS_0[A]} = \underbrace{N_\xi \det\left(\tfrac{1}{e}\partial^2\right) \int \mathcal{D}\alpha}_{\text{= "const} \times \infty\text{": does not contribute to correlation functions}} \int \mathcal{D}A \int \mathcal{D}\omega \, e^{iS_0[A]} e^{-i\int d^4x \frac{\omega^2}{2\xi}} \times$$

$$\underbrace{\times \, \delta(\partial^\mu A_\mu - \omega(x))}_{= e^{i\int d^4x \left\{ \mathcal{L}_0 - \frac{(\partial_\mu A^\mu)^2}{2\xi} \right\}}}$$

<u>bottom line</u> : can replace

$$S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) A_\nu(-k)$$

$$\longrightarrow S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k) \left(-k^2 g^{\mu\nu} + \left(1-\tfrac{1}{\xi}\right) k^\mu k^\nu\right) A_\nu(-k)$$

$\rightsquigarrow$ this allows to obtain the photon propagator $D_F^{\mu\nu}$:

$$\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right) k^{\mu\nu}\right) D_{F\,\nu\rho}(k) = i\,\delta^{\mu}_{\rho}$$

$$\Rightarrow \boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)}$$

typical choices: $\xi = 0$: "Landau gauge"

$\xi = 1$: "Feynman gauge"

$\xi = \infty$: "unitary gauge"

NB: final result (for correlation functions etc) must be independent of ξ!

⌈ $\rightsquigarrow$ only at the level of $\mathcal{M}$, not for individual diagrams! ⌋

9. Gauge invariance

consider free fermion theory : $\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$

$\Rightarrow$ global symmetry $\psi(x) \longrightarrow e^{i\alpha}\psi(x)$

Q : needed once several fields are present
$\rightsquigarrow \sum Q_i = 0$ from (effective) coupling terms

$\Rightarrow \bar{\psi}\gamma^\mu\psi$

claim : The full QED Lagrangian, including the very existence of the gauge field A^μ, follows from promoting this global symmetry to a local one, i.e. $\alpha \to \alpha(x)$!

$\equiv$ "gauge"

[NB: mass term automically satisfies this, but not the derivative]

$\rightsquigarrow$ must replace $\partial_\mu \longrightarrow D_\mu$ ("covariant derivative") such that

$$D_\mu\psi \longrightarrow e^{i\alpha(x)} D_\mu\psi$$

Ansatz : $\boxed{D_\mu = \partial_\mu + \text{"}X\text{"} = \partial_\mu + ieA_\mu(x)}$

(A_μ is called a "connection")

$$\Rightarrow D_\mu \psi \longrightarrow (\partial_\mu + ie\tilde{A}_\mu(x))\, e^{i\alpha(x)}\psi(x)$$

$$= (i(\partial_\mu\alpha) + ie\tilde{A}_\mu)e^{i\alpha}\psi + e^{i\alpha}\partial_\mu\psi$$

$$\overset{!}{=} e^{i\alpha}\underbrace{(\partial_\mu + ieA_\mu)}_{D_\mu}\psi$$

$\Rightarrow$ possible iff A_μ transforms as

$$\boxed{A_\mu \longrightarrow \tilde{A}_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha} \quad (*)$$

NB: • the same as you know from classical ED!

• we were forced to introduce A_μ

<u>next step</u>: how to construct locally invariant terms involving only A_μ (and ψ)?

• "mass terms" $A_\mu A^\mu$ are <u>not</u> invariant under $(*)$!

• consider $[D_\mu, D_\nu]\psi = \underbrace{[\partial_\mu,\partial_\nu]}_{=0} + ie\underbrace{[\partial_\mu, A_\nu]}_{(\partial_\mu A_\nu)} + ie\underbrace{[A_\mu, \partial_\nu]}_{-(\partial_\nu A_\mu)}$

$$- e^2\underbrace{[A_\mu, A_\nu]}_{0}$$

$$= ie\underbrace{\{(\partial_\mu A_\nu) - (\partial_\nu A_\mu)\}}_{= F_{\mu\nu}}$$

$$\Rightarrow \boxed{[D_\mu, D_\nu] \equiv ie F_{\mu\nu}}$$

$\rightsquigarrow$ contains no derivatives

$\Rightarrow$ commutes with $e^{i\alpha(x)}$

$\Rightarrow$ invariant under gauge transformations

[& $[\partial_\mu, \partial_\nu] = 0$]

$\Rightarrow$ all functions of $F_{\mu\nu}$ are also gauge-invariant

... $\Rightarrow$ There are only 4 possible terms to construct an invariant Lagrangian up to dimension 4 operators:

$$\boxed{\mathcal{L} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi - \frac{1}{4}(F_{\mu\nu})^2 - c\,\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}}$$

$\mathcal{L}_{QED}$!

violates P, T

$\rightarrow$ excluded by observations !

(in SM this term vanishes after field re-definitions)

[why dim 4 ?

$\rightarrow$ QFT 2 : dim > 4 operators are not "renormalizable"

$\rightsquigarrow$ cannot appear in a "fundamental" theory (valid for all energies)

How to determine dimension?

- $S = \int d^4x \, \mathcal{L}$ must be dimensionless

$$\Rightarrow [\mathcal{L}] = [\text{mass}]^4$$

$$[x^\mu] = [\text{mass}]^{-1} = [\partial_\mu]^{-1}$$

$\rightsquigarrow$
- Bosonic fields must have (mass) dim 1:

$$\text{e.g. } [(\partial\phi)^2] = [m^2\phi^2] = [F_{\mu\nu}^2] = [\text{mass}]^4$$

- Fermionic fields must have dim $3/2$:

$$\text{e.g. } [m\bar{\psi}\psi] = [\bar{\psi}\partial\!\!\!/\psi] = [\text{mass}]^4$$

$\lrcorner$

$$\mathcal{L} \supset \frac{1}{\Lambda^3}\bar{\psi}\psi F_{\mu\nu}F^{\mu\nu}$$