Yang-Mills theories

not a spinor index!

Consider "different" fermion fields ψ_i,
arrange them in multiplets:

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \qquad \text{e.g.} \begin{pmatrix} p \\ n \end{pmatrix}, \begin{pmatrix} \nu \\ e \end{pmatrix}_L, \begin{pmatrix} \tilde{W}^+ \\ \tilde{W}_0 \\ \tilde{W}^- \end{pmatrix}$$

$\Rightarrow$ free Dirac Lagrangian:

$$\mathcal{L} = \sum_i \bar{\psi}_i (i\partial - m_{(i)})\psi_i = \bar{\Psi}_i \underbrace{(i\partial - m)_{ij}}_{\text{matrix (diagonal) in multiplet indices}} \Psi_j$$

Consider any continuous group of transformations

$$\Psi \longrightarrow \Psi' = V\Psi \qquad | \; V\text{: unitary } n\times n \text{ matrix}$$

$$= \exp(i\alpha^a t^a)\,\Psi$$

real parameters

$n\times n$ matrices
"generators"
$t^{a+} = t^a \Rightarrow V^{-1} = V^+$

$$= 1 + i\alpha^a t^a + \mathcal{O}(\alpha^2)$$

$\Rightarrow \bar{\Psi}(i\partial - m)\Psi$ is invariant: global symmetry.

$\alpha^a \to \alpha^a(x) \Rightarrow \bar{\Psi}(i\not{\partial} - m)\Psi$ invariant: <u>local</u> symmetry

iff $\boxed{D_\mu \equiv \partial_\mu - i g A_\mu^a(x) t^a}$

⤳ have to introduce one vector field A_μ^a for each independent generator of the local symmetry! ($\neq n$)

check: $D_\mu \psi \longrightarrow (\partial_\mu - i g \tilde{A}_\mu^a(x) t^a) \exp[i\alpha^b(x) t^b]$

$$\overset{\alpha \ll 1}{=} (\partial_\mu - i g \hat{A}_\mu^a t^a)(1 + i\alpha^b t^b)\psi$$

$$\overset{!}{=} (1 + i\alpha^b t^b)(\partial_\mu - i g A_\mu^a t^a)\psi$$

for gauge invariance of $\bar{\Psi}\not{D}\Psi$

⤳ this fixes the transformation of A:

$$0 = \cancel{\partial_\mu} - i g \tilde{A}_\mu^a t^a + \cancel{i\alpha^b t^b \partial_\mu} + i(\partial_\mu \alpha^b) t^b + g \tilde{A}_\mu^a \alpha^b t^a t^b$$
$$\quad - \cancel{\partial_\mu} + i g A_\mu^a t^a - \cancel{i\alpha^b t^b \partial_\mu} - g A_\mu^a \alpha^b t^b t^a$$

$$\Rightarrow \underbrace{\tilde{A}_\mu^a t^a + i \tilde{A}_\mu^a \alpha^b t^a t^b}_{\hat{A}_\mu^a t^a (1 + i\alpha^b t^b)} = \frac{1}{g}(\partial_\mu \alpha^b) t^b + A_\mu^a t^a + i A_\mu^a \alpha^b t^b t^a$$

$$\Big| \times (1 - i\alpha^c t^c)$$

$$\Rightarrow \hat{A}_\mu^a t^a = \frac{1}{g}(\partial_\mu \alpha^a) t^a + A_\mu^a t^a \underbrace{- i A_\mu^a \alpha^b t^a t^b + i A_\mu^a \alpha^b t^b t^a}$$

$$i A_\mu^a \alpha^b \underbrace{[t^b, t^a]}_{\equiv f^{abc}\, t^c}$$

⌜because $t^b t^a$ $\Rightarrow [t^b, t^a]$ must be a group element⌟

f^{abc}: "structure constants":

uniquely determine (local) properties on any group!

$f^{abc} \neq 0 \iff$ "non-Abelian"

$\Rightarrow$ transformation laws for ψ, A_μ^a:

$$\boxed{\begin{aligned} \psi &\to (1 + i\alpha^a t^a)\psi \\ A_\mu^a &\to A_\mu^a + \frac{1}{g}(\partial_\mu \alpha^a) + f^{abc} A_\mu^b \alpha^c \end{aligned}}$$

$\alpha \ll 1$ $\rightsquigarrow \tilde{\psi} = \underbrace{\exp[i\alpha^a t^a]}_{V}\psi$

$\rightsquigarrow \tilde{A}_\mu^a t^a = V(A_\mu^a t^a + \frac{i}{g}\partial_\mu)V^+$ (*)

⌜(*) $(\partial_\mu - ig\tilde{A}_\mu^a t^a)V\psi = V(\partial_\mu - ig A_\mu^a t^a)\psi \quad | \; V^+ V = 1$

$$\Rightarrow \tilde{A}_\mu^a t^a V\psi = V A_\mu^a t^a V^+ V\psi$$

$$+ \frac{i}{g}\underbrace{(V\partial_\mu\psi - \partial_\mu V\psi)}$$

$$-(\partial_\mu V)\psi = -(\partial_\mu V)V^+ V\psi$$

$$= [-\partial_\mu \underbrace{(V V^+)}_{1})V + V\partial_\mu V^+]\psi$$

with $\partial_\mu(VV^+) = 0$ ⌟

field strength tensor : $\boxed{[D_\mu, D_\nu] \equiv -ig F^a_{\mu\nu} t^a}$

$\downarrow (\ldots)$

$$\boxed{\begin{aligned} F^a_{\mu\nu} &= (\partial_\mu A^a_\nu) - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ F^a_{\mu\nu} &\longrightarrow F^a_{\mu\nu} - \underbrace{f^{abc}\alpha^b F^c_{\mu\nu}}_{\Rightarrow F^a_{\mu\nu} \text{ not gauge-invariant}} \end{aligned}}$$

BUT:

$\Rightarrow (\ldots) \Rightarrow$ any globally symmetric function of $\psi, D\psi, F^a_{\mu\nu}, DF^a_{\mu\nu}$ is also locally symmetric!

$\Rightarrow$ most general gauge-invariant Lagrangian :
(up to dim-4, assuming P & T constant)

$$\boxed{\mathcal{L}^{YM} = \bar\psi i \not{D}\psi - m\bar\psi\psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

"Yang-Mills-Lagrangian"

n-component multiplets!

$a : 1, \ldots, N$ ← # generators of G = # gauge bosons

NB : interactions exclusively dictated by gauge invariance!
[i.e. only ingredients : – g (coupling strength)
– f^{abc} (symmetry group)

$\Rightarrow$ classical eq of motion:

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\mu} F^c_{\mu\nu} = -g \underbrace{\bar{\psi}\gamma_\nu t^a \psi}_{\equiv j^a_\nu}$$

"global symmetry current of fermion field"

example: SU(2) doublets

$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ SM: $\psi = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$

$t^a \to \frac{\sigma^a}{2}$ (Pauli matrices)

$$\left.\begin{aligned} [t^a, t^b] &= i f^{abc} t^c \\ [\sigma^i, \sigma^j] &= 2i\, \varepsilon^{ijk} \sigma^k \end{aligned}\right\} \Rightarrow f^{abc} = \varepsilon^{abc} \text{ describes SU(2) transformation}$$