Some basic facts about Lie groups

[recall : group $(G, \circ)$ $\equiv$ 1) $a, b \in G \Rightarrow a \circ b \in G$
set operation
2) $\exists e \in G : e \circ a = a \circ e = a \quad \forall a \in G$
"$\mathbb{1}$"
3) $\forall a \exists b : a \circ b = b \circ a = e$
"a^{-1}"
4) $\forall a, b, c \in G : (a \circ b) \circ c = a \circ (b \circ c)$]

Lie group $\equiv$ continuous group, where any element can be reached by repeated action of infinitesimal elements $g(\alpha)$:

$$g(\alpha) = \mathbb{1} + i \alpha^a T^a + \mathcal{O}(\alpha^2) \qquad (*)$$

hermitian "generators"
(in physics) $\rightsquigarrow$ g unitary !

$\Rightarrow$

$\equiv$ Lie algebra

- $[T^a, T^b] = i f^{abc} T^c$

"structure constants"
$\rightsquigarrow$ completely define group "locally" as in $(*)$

- $f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$ "Jacobi identity"

[from $[T^a, [T^b, T^c]] + \text{cycl. perm} = 0$]

Def. compact Lie group : • T^a have a finite-dim. representation
($\Leftrightarrow$ $\# T^a$ is finite)

[$\Rightarrow$ G is a finite-dim. compact manifold]

Abelian group $\equiv U(1) \equiv$ all group elements commute

$$\psi \to e^{i\alpha} \psi$$

• if one of the generators commutes with all others, it generates an independent $U(1)$:

$$\exists\, T_u \in \{T^a\} : [T_u, T_a] = 0 \quad \forall\, a$$

$$\Rightarrow \{T^a\} = T_u \oplus \{T^a / T_u\}$$

$$\Leftrightarrow G = U(1) \times (G/U(1))$$

• a group is called semi-simple $\Leftrightarrow$ contains no $U(1)$ factors

• " " " " simple $\Leftrightarrow$ cannot be divided into two mutually commuting sets of operators.

$$\Rightarrow \text{in general: } \boxed{G = U(1) \times \ldots \times U(1) \times \underbrace{\tilde{G}_1 \times \tilde{G}_2}_{\text{simple groups}} \times \ldots}$$

$$\Leftrightarrow \{T^a\} = T_u \oplus \ldots \oplus T_u \oplus \{T_1^a\} \oplus \{T_2^b\} \oplus \ldots$$

Theorem (Killing and Cartan): <u>all</u> compact, simple Lie groups belong to one of the following:

1. SU(N) : unitary transformations of N-dim. complex vectors ξ, η :

$$\left.\begin{aligned} \eta_a &\to U_{ab}\,\eta_b \\ \xi_b &\to U_{ab}\,\xi_b \end{aligned}\right\} \Rightarrow \eta_a^* \xi_a \longrightarrow \eta_a^* \xi_a$$

$$[\,\eta^\dagger \xi \longrightarrow \eta^\dagger \xi\,]$$

NB: def. excludes $U = \mathbb{1} \cdot e^{i\alpha}$!

• U unitary $\Rightarrow$ $1 = \det U = \exp[\mathrm{Tr}[\underbrace{\ln U}_{i\alpha^a T^a}]]$

$$= \exp[i\alpha^a \,\mathrm{Tr}[T^a]]$$

$$= 1 + i\alpha^a \,\mathrm{Tr}[T^a]$$

$\Rightarrow$ $\underline{\mathrm{Tr}[T^a] = 0}$

$\Rightarrow$ $\boxed{N^2 - 1}$ independent matrices

e.g. SU(2) : 3 Pauli matrices ✓

SU(3) : 8 Gell-Mann matrices (→ QCD)

2. SO(N) : orthogonal transformations of N-dim <u>real</u> vector :

$\Leftrightarrow$ $\eta_a E_{ab} \xi_b = \text{const.}$ with $E_{ab} = \delta_{ab}$

$\Leftrightarrow \; u^T u = \mathbb{1}$

$\Rightarrow \; \boxed{N(N-1)/2}$ generators ⌈1 independent rotation angle per plane in N dimensions⌋

3. $Sp(N)$: "symplectic transformations"

$\Leftrightarrow \; \eta_a E_{ab} \xi_b = \text{const.}$ with $E_{ab} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$

$\mathbb{1}$: $\frac{N}{2} \times \frac{N}{2}$ unit matrices

$\Rightarrow \; \boxed{N(N+1)/2}$ generators

4. five "exceptional" Lie algebras: $G_2, F_4, \underline{E_6}, E_7, \underline{E_8}$

($\Leftrightarrow$ "classical" 1. – 3.)

representations $\{t^a\}$

in general: $t^a \longrightarrow u t^a u^\dagger = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{\;2\;} & 0 \\ 0 & 0 & \boxed{3} \ddots \end{pmatrix}$ i.e. block-diagonal form for all t^a

- $\boxed{1}, \boxed{2}, \ldots$ are irreducible representations of group/algebra

i.e. $\{t^a\} = \{t^a_{r_1}\} \oplus \{t^a_{r_2}\} \oplus \{t^a_{r_3}\} \oplus \ldots$

standard convention: use subscript "r" to denote irreducible representations

- for each irreducible representation r

$$\phi \longrightarrow (1 + i\alpha^a t_r^a)\phi$$

there is an asociated conjugate rep. $\bar{r}$:

$$\phi^* \longrightarrow (1 - i\alpha^a (t_r^a)^*)\phi^*$$

$\underbrace{- i\alpha^a (t_r^a)^*}_{\equiv +i\alpha^a t_{\bar{r}}^a}$ ⤳ must also be an element of a rep. of the same group / algebra !

$$\Rightarrow t_{\bar{r}}^a = -(t_r^a)^* \quad | \; t = t^\dagger$$

$$= -(t_r^a)^T$$

⇒ possible to combine fields transforming in $\bar{r}, r$ to form invariants ! (like $\phi^* \phi$)

- representation is called "real" $:\Leftrightarrow t_{\bar{r}}^a = U t_r^a U^\dagger$

$\Rightarrow \exists\, G_{ab} : G_{ab}\eta_a \xi_b = \text{const.}$

for η, ξ both belonging to representation r

- $G_{ab} = G_{ba}$: r is called "strictly real" (e.g. $SO(N)$)
- $G_{ab} = -G_{ba}$: r " " "pseudo-real" (e.g. $Sp(N)$)

fundamental representation

$\equiv$ lowest-dim. irreducible representation

- $SU(N)$: N-dim. vector (complex!)

(i.e. rep. = "matrix acting on an N-dim. vector")

- $N=2$: $\varepsilon^{ab}\eta_a\eta_b = \text{const.}$

⇝ spinors are a pseudoreal rep. of $SU(2)$ [⟺ 3D vector is a real rep.! $v_a w_a = \text{const.}$]

- $N>2$: rep. is complex (not real)

$\Rightarrow \exists$ 2 inequivalent reps. $\bar{N}, N$

- $SO(N)$: N-dim real vector

→ strictly real rep.

- $Sp(N)$: N-dim vector

→ pseudo-real rep.

"adjoint representation"

(denoted w/ $r = G$)

$$(t^b_G)_{ae} \equiv i f^{abc} = -(t^b_G)^* \equiv t^{\bar{b}}_G$$

$\Rightarrow$ always a real rep.!

[why is this a representation of G?

$$[t^b_G, t^c_G]_{de} = \underbrace{\ldots}_{\text{Jacobi identity}} = i f^{abc}(t^a_G)_{de}$$
]

$$\Rightarrow \boxed{d(G) = \begin{cases} N^2-1 & \text{for } SU(N) \\ N(N-1)/2 & \text{for } SO(N) \\ N(N+1)/2 & \text{for } Sp(N) \end{cases}}$$

(= # of generators)

general field ϕ in adjoint rep.:

- $\phi_a \longrightarrow \phi_a + i\alpha^c (t^c_G)_{ab}\phi_b = \phi_a - f^{abc}\alpha^b\phi^c$

  c.f.: $F^a_{\mu\nu} \longrightarrow F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}$

  $\Rightarrow$ $F_{\mu\nu}$ belongs to adjoint representation of G!

- $D_\mu \phi_a = \partial_\mu \phi_a - ig A^b_\mu (t^b_G)_{ac}\phi_c = \partial_\mu \phi_a + g f^{abc} A^b_\mu \phi_c$

  c.f. $A^a_\mu \longrightarrow A^a_\mu + \frac{1}{g}(\partial_\mu \alpha^a) + f^{abc}A^b_\mu \alpha^c$

  $\equiv A^a_\mu + \frac{1}{g}(D_\mu \alpha^a)$

  adjoint rep.!

  $\rightsquigarrow$ classical equations of motion: $\boxed{(D^\mu F_{\mu\nu})^a = -g j^a_\nu}$

take-home for YM theories:

- gauge fields necessarily live in adjoint rep.
- rep. for fermion field is not fixed in general

  $\rightsquigarrow \bar{\psi} i \not{D} \psi$

  in Standard model: ψ multiplets (= "matter fields") are always in fundamental rep!

<u>examples</u> : • SU(2) – fundamental rep. : 2D vector ("doublet")

containing fermion fields

$\xrightarrow{\text{in SM}}$ $\psi = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$

– $d(G) = 3$ vector bosons in

adjoint rep. $\rightsquigarrow W^\pm, Z$

• SU(3) – fundamental rep : 3D vector ("triplett")

Containing fermion fields

$\hat{=}$ SM: quarks come in 3 different

"colours" : $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

– $d(G) = 8$ vector bosons

in adjoint rep. : "gluons"