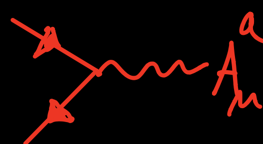
10. Quantization of non-Abelian gauge theories

$$\mathcal{L}_{YM} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar\psi\,(i\not{D}-m)\,\psi \longleftarrow \text{SM : fundamental rep.}$$

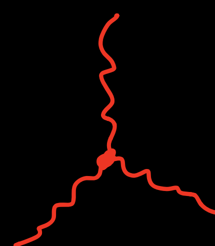
with $D_\mu = \partial_\mu - i g A^a_\mu t^a_r$

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

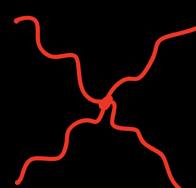
$= \mathcal{L}_0$ (quadratic terms $\sim$ propagators)

$+ g A^a_\mu t^a_r \bar\psi \gamma^\mu \psi \longrightarrow$  $= i g t^a_r \gamma^\mu$

$$\underbrace{-\frac{1}{2} g f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)}_{-g f^{abc} A^b_\mu A^c_\nu \partial^\mu A^{\nu a}} \longrightarrow$$

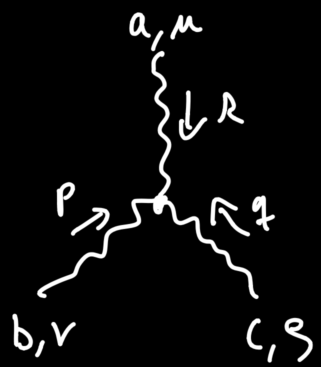


$-\frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \longrightarrow$



vector boson self-couplings

1st step : assign "external" momenta & Lorentz + gauge indices



NB: standard <u>convention</u>:

all momenta $\boxed{\text{ingoing} \Leftrightarrow k_\mu \text{ "}=\text{" } i\partial_\mu}$

[why? $\partial_\mu \phi\, |in\rangle \propto \partial_\mu \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\, a\, |in\rangle$

$\partial_\mu \to -ik_\mu$

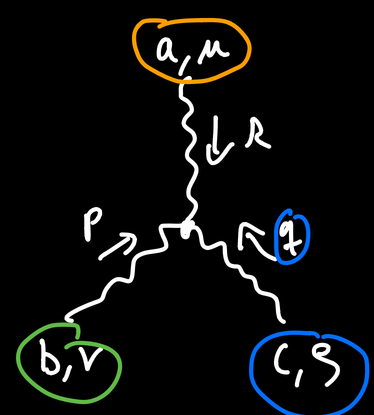
$\Leftrightarrow$ outgoing: $\langle out|\, \partial_\mu \phi \propto \langle out|\, \partial_\mu \int \frac{d^4k}{(2\pi)^4} e^{+ikx}\, a^\dagger$

$\Rightarrow \boxed{\text{outgoing} \Leftrightarrow k_\mu \text{ "}=\text{" } -i\partial_\mu}$

2^{nd} step: consider all possible contractions corresponding to this diagram:

$$\mathcal{L} \supset -g f^{\tilde a \tilde b \tilde c} A^{\tilde b}_{\tilde\mu} A^{\tilde c}_{\tilde\nu}\, \partial^{\tilde\mu} A^{\tilde\nu \tilde a} = i g f^{\tilde a \tilde b \tilde c} g_{\mu'\mu''} g_{\nu'\nu''} \tilde k^{\mu'} A^{\mu''\tilde b} A^{\nu'\tilde c} A^{\nu''\tilde a}$$

$\partial^{\tilde\mu} \to -i\tilde k^{\tilde\mu}$

$\Rightarrow$ 

$$= -g f^{\tilde a \tilde b \tilde c} g_{\mu'\mu''} g_{\nu'\nu''} \cancel{\tilde k^{\mu'}} \times$$

$$\{ \delta^{a\tilde b} \delta^{\mu''}_{\mu}\, \delta^{b\tilde c} \delta^{\nu'}_{\nu}\, \delta^{c\tilde a} \delta^{\nu''}_{\rho}\, q^{\mu'}$$

$$+ \delta^{a\tilde b} \delta^{\mu''}_{\mu}\, \delta^{c\tilde c} \delta^{\nu'}_{\rho}\, \delta^{b\tilde a} \delta^{\nu''}_{\nu}\, p^{\mu'}$$

$$+ \text{ 4 more terms} \}$$

$$= -g f^{cab} g_{\mu'\mu} g_{\nu\rho} q^{\mu'}$$

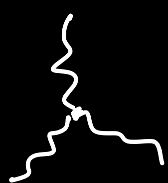
$$- g f^{bac} g_{\mu'\mu} g_{\rho\nu} p^{\mu'}$$

$+ 4$ more terms

$f^{cab} = -f^{acb} = f^{abc}$

$$= -g f^{abc} \{ g_{\nu\rho} q_\mu - g_{\nu\rho} p_\mu + 4 \text{ more terms} \}$$

$\rightsquigarrow$



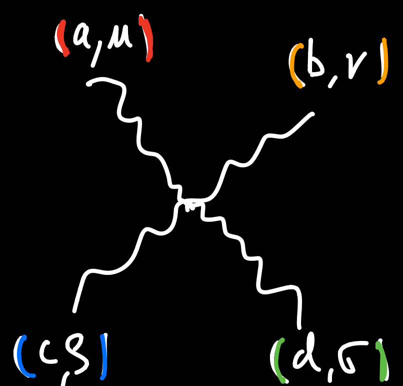
$$\boxed{= g f^{abc} \{ g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu \}}$$

4-point coupling

Same procedure:

$$\mathcal{L} \supset -\frac{1}{4} g^2 f^{abc} f^{aef} A^b_\mu A^c_\nu A^{e\mu} A^{f\nu}$$

$$= -\frac{1}{4} g^2 f^{a'b'c'} f^{a'e'f'} g^{\mu'\rho'} g^{\nu'\sigma'} \underline{A^{b'}_{\mu'}}\,\underline{A^{c'}_{\nu'}}\,\underline{A^{e'}_{\rho'}}\,\underline{A^{f'}_{\sigma'}}$$

$\Rightarrow$ 

$$= i\left(-\frac{1}{4}\right) g^2 f^{a'b'c'} f^{a'e'f'} g^{\mu'\rho'} g^{\nu'\sigma'}$$

$$\times \underbrace{\{4! = 24 \text{ terms}\}}_{\delta^a_{b'}\delta^\mu_{\mu'}\delta^b_{c'}\delta^\nu_{\nu'}\delta^c_{e'}\delta^\rho_{\rho'}\delta^d_{f'}\delta^\sigma_{\sigma'} + \delta^a_{b'}\delta^\mu_{\mu'}\delta^b_{c'}\delta^\nu_{\nu'}\delta^c_{f'}\delta^\rho_{\sigma'}\delta^d_{e'}\delta^\sigma_{\rho'} + 22 \text{ more terms}}$$

$$= -\frac{i}{4} g^2 \Big\{ f^{a'ab} f^{a'cd} g^{\mu\rho} g^{\nu\sigma} + f^{a'ab} f^{a'dc} g^{\mu\sigma} g^{\nu\rho} + \underbrace{22 \text{ more terms}}_{\text{from all permutations}} \Big\}$$

$g\ (a,\mu) \leftrightarrow (b,\nu) \leftrightarrow (c,\rho) \leftrightarrow (d,\sigma)$

sets of 4 are always equal, e.g.

$$f^{a'ab} f^{a'cd} g^{\mu\rho} g^{\nu\sigma} \overset{a\leftrightarrow b,\ c\leftrightarrow d}{=} f^{a'ba} f^{a'dc} g^{\nu\sigma} g^{\mu\rho}$$

$$= f^{a'cd} f^{a'ab} g^{\nu\sigma} g^{\mu\rho}$$

$$= f^{a'dc} f^{a'ba} g^{\mu\rho} g^{\nu\rho}$$

$$\boxed{\begin{aligned} = -ig^2 \Big\{ & f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) \\ & + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) \\ & + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}\right) \Big\} \end{aligned}}$$