II. Spontaneous symmetry breaking

3 steps : i) discrete symmetry : concept of SSB
ii) cont. " : appearance of Goldstone bosons
iii) gauge " : gauge bosons [+ fermions] acquire mass ⇝ Higgs mechanism

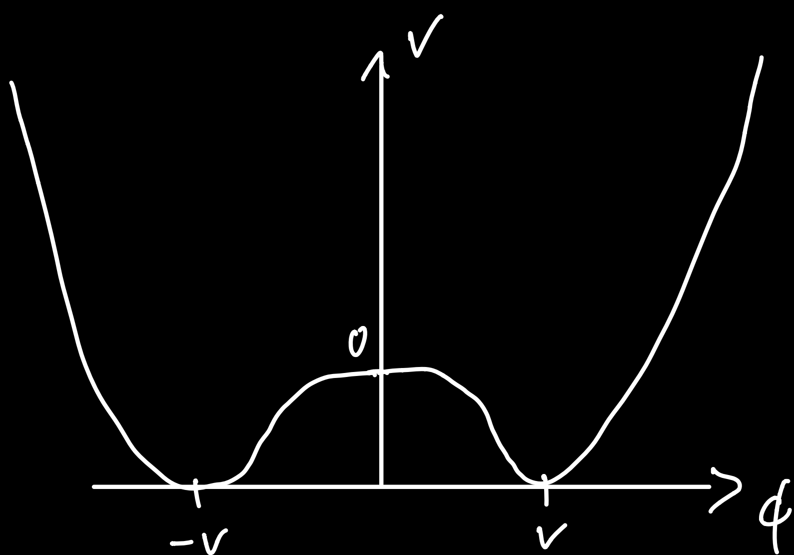
i) discrete symmetry

consider "ϕ^4" theory ⟶ real scalar ϕ, with potential $V(\phi)$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$$

choose : $V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$

$$\Rightarrow \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$



$\Rightarrow$ theory symmetric under $\phi \leftrightarrow -\phi$ $\boxed{\text{"}\mathbb{Z}_2 \text{ symmetry"}}$

minimum energy classical configuration:

$\hat{=}$ "vacuum"! $\longrightarrow$ vacuum expectation value (VEV)

$$V' \overset{!}{=} 0 \Rightarrow -\mu^2\phi_0 + \frac{\lambda}{3!}\phi^3 \overset{!}{=} 0 \Leftrightarrow \phi_0 = \pm v \equiv \pm\sqrt{\frac{6}{\lambda}}\,\mu$$

"$\langle\phi\rangle$"

$\Rightarrow$ field will be close to one of these minima, e.g. $\phi_0 = v$!

$\rightsquigarrow$ redefine $\phi(x) \equiv v + \sigma(x)$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}\mu^2(v+\sigma)^2 - \frac{\lambda}{4!}(v+\sigma)^4$$

$$= \cancel{\text{const.}} + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\,\mu\,\underline{\underline{\sigma^3}} - \frac{\lambda}{4!}\sigma^4$$

$\Rightarrow$ describes a real scalar field (σ) with

- $m_\sigma = \sqrt{2}\,\mu$
- <u>no</u> symmetry under $\sigma \to -\sigma$

i.e. "$\mathbb{Z}_2$ symmetry is <u>spontaneously broken</u>"

ii) continuous symmetry

consider $\underline{\underline{2}}$ real scalar fields, ϕ^1 & ϕ^2

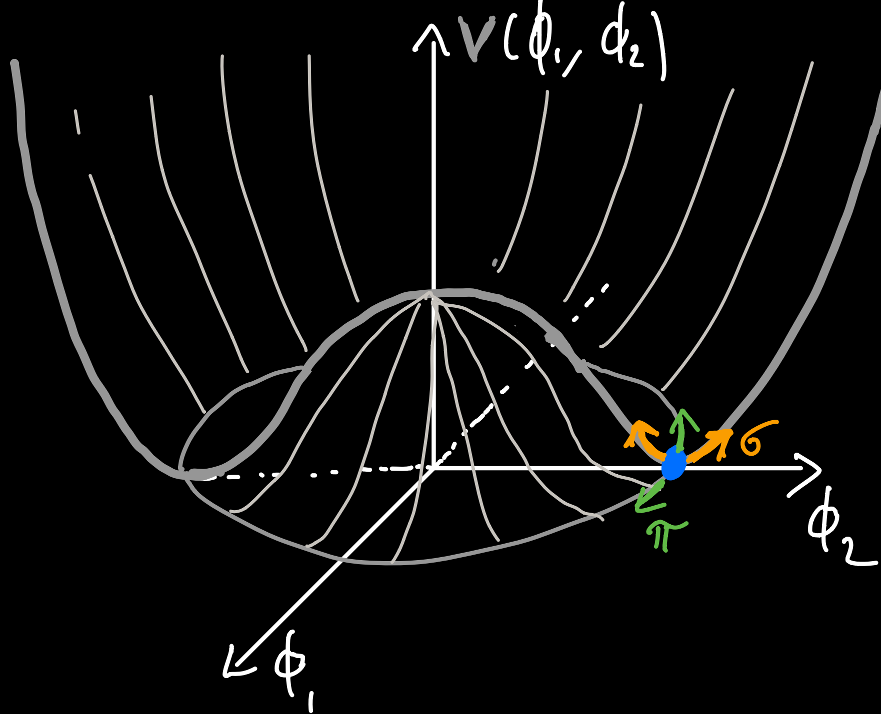
$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \underbrace{\frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}\left[(\phi^i)^2\right]^2}_{-V(\phi^i)} \qquad \Big|\ (\phi^i)^2 = (\phi^1)^2 + (\phi^2)^2$$

$\Rightarrow$ symmetric under rotations, $SO(2)$:

$$\phi^i \longrightarrow R^{ij}\phi^j \equiv \phi^{i'} \qquad w/\ R^T = R^{-1}, \text{ i.e. } R^T R = \mathbb{1}$$

$$\Rightarrow (\phi^i)^2 \longrightarrow \underbrace{\underbrace{R^{ij}}_{R^{ji}}R^{ij'}}_{\delta^{jj'}}\phi^j\phi^{j'} = (\phi^j)^2$$

$$2D: R = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$



"Mexican hat potential"

minimum $\frac{\partial V}{\partial \phi^1} = \frac{\partial V}{\partial \phi^2} = 0 \iff \frac{\partial V}{\partial \phi^i} = 0$

$$\iff -\mu^2 \phi_0^i + \lambda \phi_0^i \cdot (\phi_0^i)^2$$

$$\Rightarrow (\phi_0^i)^2 = \frac{\mu^2}{\lambda} \equiv v^2 \equiv \langle \phi \rangle^2$$

i.e. only length of vector ϕ^i is determined, direction is arbitrary!

now want – again – to expand around minimum energy classical configuration!

choose e.g. $\phi_0^i = (0, v)$

introduce shifted fields : $\phi^i(x) \equiv (\pi(x), v + \sigma(x))$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}\mu^2(\pi^2 + (v+\sigma)^2) - \frac{\lambda}{4}(\pi^2 + (v+\sigma)^2)^2$$

$\vdots$

$$= \underbrace{\frac{1}{2}(\partial_\mu \pi)^2}_{\text{massless scalar field}} + \underbrace{\frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu)^2 \sigma^2}_{\text{massive scalar field w/ } m_\sigma = \sqrt{2}\mu}$$

$$\underbrace{- \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\pi^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\pi^2\sigma^2 - \frac{\lambda}{4}\pi^4}_{\mathcal{L}_{int}(\sigma, \pi)}$$

generalization:

$\boxed{\begin{array}{l}\text{Goldstone's theorem} \\ \text{For every spontaneously broken continuous} \\ \text{symmetry, the theory must contain a} \\ \text{massless particle (= "Goldstone boson")}\end{array}}$

remarks:
- Goldstone bosons remain exactly massless even when taking into account quantum corrections
- application to many light bosons in physics, e.g. pions!
- $\boxed{\begin{array}{l}\#\text{ Goldstones} = \#\text{ cont. symmetries before SSB} \\ \qquad\qquad\quad - \# \quad " \quad " \quad \text{after SSB}\end{array}}$

e.g. $i = 1, 2 \to N$ in above example
("linear sigma model")

$\Rightarrow$ breaking $SO(N) \to SO(N-1)$

$\Rightarrow \#$ Goldstones $= \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2}$

$= N-1$ fields π^i