Introduction to quantum field theory (FYS 4170)

recall : quantization à la Schrödinger
w/ • relativistic energy-momentum relation
• number N of particles is conserved

=> 1. concept of particles ?
2. violation of causality
3. negative energy states
[4. spin ?]
5. probability interpretation !?

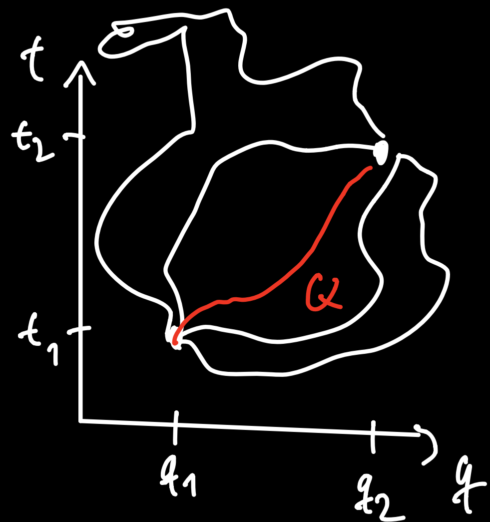
⤳ try to construct a multiparticle ("field") theory!

N not conserved

1. Classical field theory

Lagrangian and Hamiltonian

point particle in 1D : action $S = \int dt\, L[q(t), \dot{q}(t)]$



Q : physical trajectory / "classical path"

⇝ satisfies

$$\delta S \overset{!}{=} 0 \iff \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0}$$

(⇝ solution: $q(t) = Q(t)$)

1D → N dimensions: $q \to \mathbf{q} = (q_1, \dots, q_N)$

$$\dot{\mathbf{q}} = \frac{d}{dt}\mathbf{q}$$

field theory: consider a Lagrangian density instead
[required by locality!]

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

"Lagrangian" $\phi(x^\mu)$ "field"

$\hat{=}\, q_i$

principle of least action:

$$0 \overset{!}{=} \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \underbrace{\delta(\partial_\mu \phi)}_{\partial_\mu(\delta\phi)} \right\}$$

NB: sum convention!

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right)} - \delta\phi\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \right\}$$

$$\to \int_{\partial V} d\Sigma_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi$$

$\to 0$ if i) $\delta\phi(t_1, \vec{x}) = \delta\phi(t_2, \vec{x}) = 0$

ii) $\delta\phi(t_1 < t < t_2, \vec{x}) \to 0$ sufficiently fast for $|\vec{x}| \to \infty$

$$\Rightarrow 0 \overset{!}{=} \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \right\} \delta\phi \quad \underline{\underline{\forall\, \delta\phi(x^\mu)}}\ !$$

$$\Rightarrow \boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$ "Euler Lagrange equations"

$\rightsquigarrow$ equations of motion for $\phi(x^\mu)$

NB: straightforward to describe multiple fields: $\phi \to \phi_a$

recall: $p \equiv \frac{\partial L}{\partial \dot{q}}$ (in classical mechanics)

$\rightsquigarrow$ canonical (!) momentum <u>density</u> $$\boxed{\pi(x^\mu) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}}$$

$\Rightarrow$ "Hamiltonian" $\boxed{H \equiv \int d^3x \left\{ \underbrace{\pi(x^\mu)\dot{\phi}(x^\mu) - \mathcal{L}}_{\equiv \mathcal{H} = \mathcal{H}(\phi, \pi, \vec{\nabla}\phi)} \right\}}$

<u>example</u> : scalar field $\phi(x^\mu)$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\underbrace{(\partial_\mu \phi)^2}_{(\partial_\mu\phi)(\partial^\mu\phi)} - V(\phi)$$

$V(\phi) \rightarrow \frac{1}{2}m^2\phi^2$ = "1st term in Taylor expansion"

"=" $T - V$

$$\Rightarrow \cdot \frac{\partial \mathcal{L}}{\partial \phi} = -V' \rightarrow -m^2\phi$$

$$\cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\phi)}\left[(\partial_\nu\phi)(\partial_\sigma\phi)\eta^{\nu\sigma}\right]$$

$$\eta^{\nu\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\left[\underbrace{\delta^\mu_\nu(\partial_\sigma\phi)\eta^{\nu\sigma} + (\partial_\nu\phi)\delta^\mu_\sigma\eta^{\nu\sigma}}_{\partial^\mu\phi}\right]$$

$$= \partial^\mu\phi$$

(e.o.m.) $\Rightarrow$ $\boxed{(\underbrace{\partial^\mu\partial_\mu}_{=\Box} + m^2)\phi = 0}$ "Klein-Gordon equation" (for a <u>classical</u> field)

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi \Rightarrow \mathcal{H} = \pi\dot\phi - \mathcal{L}$$

$$= \underbrace{\tfrac{1}{2}\dot\phi^2}_{\text{"kinetic energy"}} + \underbrace{\tfrac{1}{2}(\vec\nabla\phi)^2}_{\text{"shearing in space"}} + \underbrace{\tfrac{1}{2}m^2\phi^2}_{\text{"having the field around at all"}} > 0 \checkmark$$

Noether's theorem

"for every symmetry there is a conservation law"

consider a continuous field transformation of a physical field

$$(*) \quad \phi(x) \longrightarrow \phi'(x) \equiv \phi(x) + \underbrace{(\alpha)}_{\text{small parameter}} \Delta\phi(x)$$

$$\Rightarrow \mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + (\alpha)\Delta\mathcal{L}$$

(*) is called a "**symmetry**" iff the equations of motion do not change under (*)

- sufficient condition : $\Delta\mathcal{L} = \partial_\mu J^\mu(x)$

[not using equations of motion]

• in general : $\mathcal{L} \to \mathcal{L}' = \mathcal{L}(\phi', \partial_\mu \phi')$

[using e.o.m.]

$$= \mathcal{L}(\phi, \partial_\mu \phi) + \underbrace{\alpha \frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \alpha \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \overbrace{\Delta \partial_\mu \phi}^{\partial_\mu \Delta\phi}}_{= \alpha \Delta \mathcal{L}} + O(\alpha^2)$$

$$= \alpha \left\{ \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right)}_{= 0 \text{ (e.o.m.) !}} \Delta\phi + \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right\}$$

$\Rightarrow$ if there is a symmetry (i.e. J^μ exists), then

$$\boxed{\partial_\mu \underbrace{\left(J^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right)}_{\equiv j^\mu_N \text{ "Noether current"}} = 0}$$

$\Rightarrow$ "For each symmetry $\Delta\phi$, there is a conserved current j^μ_N (and conserved charge $Q = \int d^3x\, j^0_N$)"

application : the energy-momentum tensor

space-time translations : $x^\nu \longrightarrow x'^\nu = x^\nu + a^\nu$

NB: 4 different symmetries ($\nu = 0, 1, 2, 3$)

$$\Rightarrow \phi(x) \longrightarrow \phi'(x) = \phi(x+a) = \phi(x) + a^\nu \underbrace{\partial_\nu \phi(x)}_{\equiv (\Delta\phi)_\nu} + \mathcal{O}(a^2)$$

$$\Rightarrow \mathcal{L} \longrightarrow \mathcal{L} + a^\nu \partial_\nu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \underbrace{(\delta^\mu_\nu \mathcal{L})}_{\equiv (J^\mu)_\nu}$$

$\Rightarrow$ four conserved currents

$$(j^\mu)_\nu \equiv \boxed{T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu}$$

$\Rightarrow$ conserved charges : i) [$\nu=0$] $\quad H = \int T^{00} d^3x = \int \mathcal{H}\, d^3x$

ii) [$\nu=i$] $\quad P^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi \, d^3x$

$\rightsquigarrow$ physical momentum !

($\leftrightarrow$ conjugate/canonical momentum π)

The Lorentz group

consider $x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$

Λ is a Lorentz transformation

$\Leftrightarrow \; x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu \overset{!}{=} x'^2$

$\Leftrightarrow \; x^T \eta x \overset{!}{=} (x')^T \eta x' = (\Lambda x)^T \eta (\Lambda x)$

$$= x^T \Lambda^T \eta \Lambda x$$

$\Leftrightarrow \; \boxed{\eta = \Lambda^T \eta \Lambda} \quad (\#)$

$\left(\Leftrightarrow \; \eta_{\mu\nu} = \Lambda^\tau{}_\mu \eta_{\tau\sigma} \Lambda^\sigma{}_\nu \right)$

c.f. 3D rotations : $x^i \longrightarrow R^{ij} x^j$

R is a rotation $\Leftrightarrow \; x^2 \equiv \vec{x} \cdot \vec{x} = \text{const.}$

$\Leftrightarrow \; \mathbb{1}_{3\times3} = R^T \mathbb{1}_{3\times3} R$

$\Leftrightarrow$ "R is orthogonal"

- LTs form a group L : $\{\{\Lambda\}, \cdot\}$
  - $\Lambda_1, \Lambda_2 \in L \Rightarrow \Lambda_1 \cdot \Lambda_2 \in L$ "closure"
  - $\forall \Lambda_1, \Lambda_2, \Lambda_3 \in L : (\Lambda_1 \cdot \Lambda_2) \cdot \Lambda_3 = \Lambda_1 \cdot (\Lambda_2 \cdot \Lambda_3)$ "associativity"

- $\forall \Lambda \in L : \underset{\underset{L}{\uparrow}}{\underline{\mathbb{1}}}\Lambda = \Lambda \cdot \mathbb{1} = \Lambda$ "identity"
- $\forall \Lambda \in L \; \exists \Lambda^{-1} \in L : \Lambda \cdot \Lambda^{-1} = \Lambda^{-1}\Lambda = \underline{\mathbb{1}}$ "inverse"

but <u>not</u> Abelian, i.e. in general $\Lambda_1 \cdot \Lambda_2 \neq \Lambda_2 \cdot \Lambda_1$

<u>decomposition</u> of <u>Lorentz</u> gro<u>u</u>p <u>L</u>

(*) $\Rightarrow$ i) $\det \eta = \det \Lambda \cdot \det \eta \cdot \det \Lambda$

$\Rightarrow \det \Lambda = \pm 1$

ii) $\eta_{00} = 1 = \eta_{\rho\sigma} \Lambda^{\rho}{}_{0} \Lambda^{\sigma}{}_{0} = (\Lambda^{0}{}_{0})^2 - (\Lambda^{1}{}_{0})^2 - (\Lambda^{2}{}_{0})^2 - (\Lambda^{3}{}_{0})^2$

$\Rightarrow (\Lambda^{0}{}_{0})^2 \geq 1$

$\Rightarrow \Lambda^{0}{}_{0} \; \begin{matrix} \geq 1 \\ \leq -1 \end{matrix}$

$\Rightarrow L$ splits into 4 disconnected subsets:

$L = L_+^{\uparrow} \cup L_+^{\downarrow} \cup L_-^{\uparrow} \cup L_-^{\downarrow}$ $\quad \pm : \det \Lambda = \pm 1$

$\uparrow\downarrow : \Lambda^{0}{}_{0} \; \begin{matrix} \geq +1 \\ \leq -1 \end{matrix}$

group of proper Lorentz trafos

$\Rightarrow$ general LT : $\boxed{\Lambda = P^m T^n \Lambda_0}$; $m, n \in \{0, 1\}$

where $\Lambda_0 \in L_+^\uparrow$: boosts and rotations

$$P : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{"spatial reflection"}$$

$$T : \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{"temporal reflection"}$$

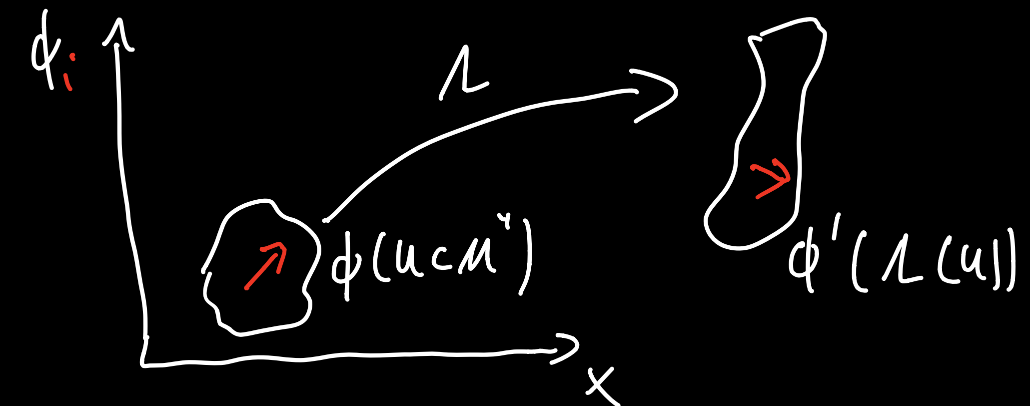
relativistic invariance

an expression is "relativistically invariant" if it takes the same form

a) in all frames of reference ("passive" point of view)

b) after boosting/rotating all fields ("active" —)

$\rightsquigarrow$ both are equivalent! — we will adopt this one!



$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\Rightarrow \boxed{\begin{aligned} \phi(x) &\to \phi'(x) = \phi(\Lambda^{-1}x) \\ V^\mu(x) &\to V'^\mu(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x) \end{aligned}}$$

"scalar" field

"4-vector" field

NB: $\mathcal{L}$ is a Lorentz scalar

$\Rightarrow$ equations of motion are automatically relativistically invariant!

The Lorentz algebra

motivation: how to construct Lorentz-invariant equations?

1st guess: "count indices" [each term must have the same set of uncontracted indices]

$\rightsquigarrow$ problem: this gives only a subset of all possibilities!

more general: find all possible transformation laws for an N-component field $\phi_a(x)$

$a = 1 \dots N$; not a space-time index

Solution : • let's restrict ourselves to infinitesimal transformations [$\Lambda^\uparrow_+$ is a continuous group!]

$$\Rightarrow \phi_a(x) \xrightarrow{LT} \phi'_a = M_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x)$$

$N \times N$ matrix

• only requirement on M: preserve correspondence between M & Λ for subsequent LT's (i.e. M must be a representation of L!)

$$\Leftrightarrow \boxed{\Lambda'' = \Lambda'\Lambda \Rightarrow M_{ab}(\Lambda'') = M_{ac}(\Lambda')\,M_{cb}(\Lambda)} \quad (*)$$

$\rightarrow$ now find all solutions to this!

i) we consider infinitesimal transformations:

$$\boxed{\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,, \quad |\omega^\mu{}_\nu| \ll 1\,, \quad \omega_{\mu\nu} \overset{(*)}{=} -\omega_{\nu\mu}}$$

$$\Rightarrow \boxed{M_{ab}(\Lambda) = \delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})_{ab} + \dots\,; \quad J^{\mu\nu} = -J^{\nu\mu}}$$

(convention!)

┌ why $\omega_{\mu\nu} = -\omega_{\nu\mu}$? expand $\eta = \Lambda^T \eta \Lambda$!

$$\Leftrightarrow \eta_{\mu\nu} = \eta_{\rho\sigma}(\delta^\rho{}_\mu + \omega^\rho{}_\mu)(\delta^\sigma{}_\nu + \omega^\sigma{}_\nu)$$

$$= \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + \mathcal{O}(\omega^2)$$

⊡ ┘

ii) apply (*) to $\Lambda'' = \Lambda \Lambda' \Lambda^{-1}$

┌ inf. Λ'

$$\Rightarrow M(\Lambda(\mathbb{1}+\omega')\Lambda^{-1}) \overset{!}{=} M(\Lambda)\, M(\Lambda' = \mathbb{1}+\omega')\, \underbrace{M(\Lambda^{-1})}_{= M^{-1}(\Lambda)}$$

only $\mathcal{O}(\omega')$

$$\Leftrightarrow \frac{i}{2} \underbrace{(\Lambda \omega' \Lambda^{-1})_{\mu\nu}}_{\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \omega'_{\rho\sigma}} J^{\mu\nu} = M(\Lambda)\left(\frac{i}{2}\omega'_{\mu\nu} J^{\mu\nu}\right) M^{-1}(\Lambda)$$

| $\eta = \Lambda^T \eta \Lambda$
$\Rightarrow \eta \Lambda^{-1} = \Lambda^T \eta$
$\Rightarrow \Lambda^{-1} \text{"="} \Lambda^T$

$$\Leftrightarrow \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma J^{\mu\nu} = M(\Lambda) J^{\rho\sigma} M^{-1}(\Lambda)$$

inf $\Lambda = \mathbb{1} + \omega$

$$\Leftrightarrow \underbrace{\omega^\rho{}_\mu J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu}}_{} = \left(\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right) J^{\rho\sigma} \underset{M^{-1}}{-} J^{\rho\sigma}\left(\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right)$$

$$= \omega_{\mu\nu}\left\{\eta^{\nu\rho} J^{\mu\sigma} \underset{\omega_{\nu\mu}}{-} \eta^{\mu\sigma} J^{\rho\nu}\right\}$$

| use that $J^{\mu\nu} = -J^{\nu\mu}$! ┘

"Lorentz algebra"

$$\Leftrightarrow \boxed{-i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\sigma} J^{\rho\mu} - \eta^{\mu\sigma} J^{\rho\nu} + \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma}}$$

$\rightarrow$ 6 "generators" $J^{\mu\nu}$ $(= - J^{\nu\mu})$

$\Vert$ $\quad$ $\uparrow$ $N \times N$ matrices

3+3 !

"boosts" + "rotations"

<u>examples</u> : 1) $J^{\mu\nu} \equiv x^\mu \hat{p}^\nu - x^\nu \hat{p}^\mu$

$$= i(x^\mu \partial^\nu - x^\nu \partial^\mu) \qquad | \; \partial_\mu = (\partial_t, \vec{\nabla}) ; \partial^\mu = (\partial_t, -\vec{\nabla})$$

in 3D : $J^{ij} = -i(x^i \nabla^j - x^j \nabla^i) \qquad | \; J^1 \equiv J^{23} ; J^2 \equiv J^{31} ; J^3 \equiv J^{12}$

$$\Leftrightarrow J^i = \varepsilon^{ijk} x^j (-i \nabla^k)$$

$$\Leftrightarrow \vec{J} = \vec{x} \times \vec{p}$$

$$\Rightarrow [J^i, J^j] = i \varepsilon^{ijk} J^k$$

- from QM : angular momentum operators
- today : these J^i form a subset of the Lorentz algebra !

2) Consider now 4x4 matrices $\tilde{J}^{\mu\nu}$, with

$$\boxed{(\tilde{J}^{\mu\nu})_{\alpha\beta} \equiv i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)}$$

$\Rightarrow$ these are the matrices that generate Lorentz transformations acting on ordinary 4-vectors !

• infinitesimal: $V^\alpha \longrightarrow \Big(\delta^\alpha_\beta \underbrace{- \frac{i}{2}\tilde{\omega}_{\mu\nu}(\tilde{J}^{\mu\nu})^\alpha_\beta}_{\equiv \omega^\alpha{}_\beta}\Big) V^\beta$

• finite: $V^\alpha \longrightarrow \Lambda^\alpha{}_\beta V^\beta$;

$$\boxed{\Lambda^\alpha{}_\beta = e^{-\frac{i}{2}\tilde{\omega}_{\mu\nu}\tilde{J}^{\mu\nu}}}$$

recall $\exp[x] = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

e.g. • $\tilde{\omega}_{12} = -\tilde{\omega}_{21} \equiv \theta$ (all remaining $\omega_{\mu\nu} = 0$)

$$\Rightarrow (\tilde{J}^{12})^\alpha{}_\beta = \eta^{\alpha\gamma}(\tilde{J}^{12})_{\gamma\beta} = -i\begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&+1&0&0\\ 0&0&0&0\end{pmatrix} = -(\tilde{J}^{21})^\alpha{}_\beta$$

$$\Rightarrow V \rightarrow \begin{pmatrix} 1&0&0&0\\ 0&1&-\theta&0\\ 0&\theta&1&0\\ 0&0&0&1\end{pmatrix} V$$: inf. rotation around z-axis!

⌜exercise: finite θ⌟

• $\tilde{\omega}_{01} = -\tilde{\omega}_{10} \equiv \eta$ "rapidity"

$$\Rightarrow V \rightarrow \begin{pmatrix} 1&\beta&0&0\\ \beta&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}$$: inf. boost in x direction!

⌜exercises: derive finite form⌟

2. The Klein-Gordon field

classical real scalar field (free)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \Rightarrow$$

- $(\partial^2 + m^2)\phi = 0$ KGE
- $\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$ $\quad \big| \ \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2$$

quantization

QM : $q_i, p_i \longrightarrow$ **operators** $\hat{q}_i, \hat{p}_i$

classical coordinates/ phase-space variables

with $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$

$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

NB: **Schrödinger** picture

$\rightarrow$ no t-dependence of operators

QFT : $\phi, \pi \longrightarrow \overset{(\wedge)}{\phi}, \overset{(\wedge)}{\pi}$

classical fields

at some **fixed** value $t = t_0$

with $\boxed{\begin{aligned} &[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^{(3)}(\vec{x} - \vec{y}) \\ &[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \end{aligned}}$

"equal time commutation relations"

energy spectrum

Fourier transform only w.r.t. $\vec{x}$ ("keep t fixed"/ t-dependence explicit)

$$\phi([t],\vec{x}) = \int \frac{d^3p}{(2\pi)} e^{i\vec{p}\vec{x}} \phi([t,]\vec{p})$$

$$\Rightarrow \text{KGE}: \boxed{\left(\frac{\partial^2}{\partial t^2} + \underbrace{\vec{p}^2 + m}_{\equiv \omega_p^2}\right)\phi = 0}$$

$\rightsquigarrow$ harmonic oscillator! ($\rightsquigarrow \phi \sim e^{\pm i\omega_p t}$)

⌈recall from QM: $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$ | $x \equiv \frac{1}{\sqrt{2\omega}}(a + a^+)$

$p \equiv -i\sqrt{\frac{\omega}{2}}(a - a^+)$

$= \omega(a^+ a + \frac{1}{2})$

$\overset{[x,p]}{\Rightarrow} [a, a^+] = 1$ ⌋

define "ladder operators"

$$\hat{\phi}(t,\vec{x}) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_{\vec{p}}\, e^{i\vec{p}\vec{x}} + a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\vec{x}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \underbrace{\frac{1}{\sqrt{2\omega_p}}\left(\hat{a}_{\vec{p}} + a^{\dagger}_{-\vec{p}}\right)}_{\phi(t,\vec{p})}$$

$$\pi(t,\vec{x}) \overset{(\dot{\phi})}{\equiv} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} (-i)\sqrt{\frac{\omega_p}{2}} \left(a_{\vec{p}} - a^{\dagger}_{-\vec{p}}\right)$$

$$\Leftrightarrow \quad a_{\vec p} = \sqrt{\frac{\omega_p}{2}}\left(\phi(\vec p) + \frac{i}{\omega_p}\tilde{\pi}(\vec p)\right)$$

$$a^{\dagger}_{\vec p} = \sqrt{\frac{\omega_p}{2}}\left(\phi(-\vec p) - \frac{i}{\omega_p}\pi(-\vec p)\right)$$

$$\Rightarrow [a_p, a^{\dagger}_{\vec p'}] = \frac{1}{2}\sqrt{\omega_p\omega_{p'}}\left[\phi(\vec p) + \frac{i}{\omega_p}\pi(\vec p), \phi(-\vec p') - \frac{i}{\omega_{p'}}\pi(-\vec p')\right]$$

$$= \frac{1}{2}\sqrt{\omega_p\omega_{p'}}\int d^3x \int d^3y \, e^{-i\vec p\vec x}\, e^{+i\vec p'\vec y} \times$$

$$\times \underbrace{\left[\phi(\vec x) + \frac{i}{\omega_p}\pi(\vec x), \phi(\vec y) - \frac{i}{\omega_{p'}}\pi(\vec y)\right]}_{= -\frac{i}{\omega_{p'}}[\phi(\vec x),\pi(\vec y)] + \frac{i}{\omega_p}\underbrace{[\pi(\vec x),\phi(\vec y)]}_{-[\phi(\vec y),\pi(\vec x)]}}$$

$$= \delta^{(3)}(\vec x - \vec y)\left(\frac{1}{\omega_p} + \frac{1}{\omega_{p'}}\right)$$

$$= \frac{1}{2}\sqrt{\omega_p\omega_{p'}}\underbrace{\int d^3x \, e^{-i\vec x(\vec p - \vec p')}}_{(2\pi)^3\delta^{(3)}(\vec p - \vec p') \;\Rightarrow\; \omega_p = \omega_{p'}}\left(\frac{1}{\omega_p} + \frac{1}{\omega_{p'}}\right)$$

$$\Rightarrow \boxed{[a_{\vec p}, a^{\dagger}_{\vec p'}] = (2\pi)^3\delta^{(3)}(\vec p - \vec p')}$$

similar: $[a_{\vec p}, a_{\vec p'}] = [a^{\dagger}_{\vec p}, a^{\dagger}_{\vec p'}] = 0$

$$\Rightarrow H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}$$

$$= \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} \underbrace{e^{i\vec{x}(\vec{p}+\vec{p}')}}_{\to (2\pi)^3 \delta^{(3)}(\vec{p}+\vec{p}')}$$

$$\times \left\{ -\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}{4}(a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{\vec{p}'} - a^\dagger_{-\vec{p}'}) + \frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})(a_{\vec{p}'} + a^\dagger_{-\vec{p}'}) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ -\frac{\omega_p}{4}(a_{\vec{p}} - a^\dagger_{-\vec{p}})(a_{-\vec{p}} - a^\dagger_{\vec{p}}) + \underbrace{\frac{\vec{p}^2 + m^2}{4\omega_p}}_{\frac{\omega_p}{4}}(a_{\vec{p}} + a^\dagger_{-\vec{p}})(a_{-\vec{p}} + a^\dagger_{\vec{p}}) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{4} 2 \left\{ a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\cancel{-\vec{p}}\,\vec{p}}\, a_{\cancel{-\vec{p}}\,\vec{p}} \right\}$$

$(\int d^3p \to \int d^3\tilde{p};\ \tilde{p} = -p)$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left\{ a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}\underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}}]}_{\propto\, \delta^{(3)}(0) = \infty} \right\}$$

= sum over all zero-point energies $\frac{\omega_{\vec{p}}}{2}$

<u>BUT</u>: experimentally we only measure <u>differences</u> to ground state energy!

$\rightsquigarrow$ ignore ... [NB: not possible in GR ... !]

similar: $\underline{P} = -\int d^3x\, \pi(\vec{x}) \nabla \phi(\vec{x}) = \ldots = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$

Def. <u>vacuum</u>: • $\langle 0|0\rangle = 1$

• $a_{\vec{p}}|0\rangle = 0 \quad \forall \vec{p}$

$\Rightarrow H|0\rangle = 0$; i.e $E = 0$

<u>particle interpretation</u>

$[\underbrace{H}_{\underline{P}}, a^\dagger_{\vec{p}}] = \underbrace{\omega_{\vec{p}}}_{\vec{p}} a^\dagger_{\vec{p}}$

$[\underbrace{H}_{\underline{P}}, a_{\vec{p}}] = -\underbrace{\omega_{\vec{p}}}_{\vec{p}} a_{\vec{p}}$

$\Rightarrow \ldots$

all energy eigenstates can be written as

$a^\dagger_{\vec{p}} \ldots a^\dagger_{\vec{q}} |0\rangle$

with energy $E = \omega_{\vec{p}} + \ldots + \omega_{\vec{q}}$

and momentum $\underline{P} = \vec{p} + \ldots + \vec{q}$

$\Rightarrow a^\dagger_{\vec{p}}$ creates an excitation with • energy $\omega_{\vec{p}} = +\sqrt{\vec{p}^2 + m^2}$ > 0!

• momentum $\vec{p}$

$\rightsquigarrow$ "particles"! (NB: discrete, but not necessarily localized in space!)

- statistics : i) $a^\dagger_{\vec p}\, a^\dagger_{\vec q}\, |0\rangle = +a^\dagger_{\vec q}\, a^\dagger_{\vec p}\, |0\rangle$

  ii) $(a^\dagger_{\vec p})^n |0\rangle \neq 0 \quad \forall\, n \geq 0$

  $\Rightarrow$ Klein-Gordon particles obey Bose-Einstein statistics!

- conventions : $|\vec p\rangle \equiv \sqrt{2E_{\vec p}}\; a^\dagger_{\vec p}\, |0\rangle$

  $$\Rightarrow \langle \vec q\, | \vec p\, \rangle = 2E_{\vec p}\, (2\pi)^3\, \delta^{(3)}(\vec p - \vec q\,)$$

  $$= \text{Lorentz invariant!}$$

  $$\left[\int \frac{d^4p}{(2\pi)^4}\, \underbrace{\delta(p^2 - m^2)}_{\delta((p_0+E_p)(p_0-E_p))}\, \theta(p^0) = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2E_p} \right]$$

  $\rightsquigarrow$ see P&S, p 22/23

- interpretation $\phi(\vec x)\,|0\rangle$

  i) $$\phi(\vec x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\, \underbrace{\frac{1}{2E_{\vec p}}}_{\to 1 \text{ in NR limit}}\, e^{-i\vec p \vec x}\, |\vec p\,\rangle = \int \frac{d^4p}{(2\pi)^4}\, \delta(p^2-m^2)\, \theta(p^0)\, e^{-i\vec p \vec x}\, |\vec p\,\rangle$$

$\Rightarrow$ recover NR / QM expression for $|x\rangle$

ii) $$\langle 0|\phi(x)|p\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \underbrace{\langle 0|(a_{\vec{p}'} + \cancel{a^\dagger_{-\vec{p}'}})\, a^\dagger_{\vec{p}}|0\rangle}_{\rightarrow (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')} \sqrt{2E_{\vec{p}}}\, e^{i\vec{p}x}$$

$$= e^{i\vec{p}\vec{x}}$$

$\propto \langle x|p\rangle$ in NR QM

$\Rightarrow$ "$\phi(\vec{x})$ creates a particle at position $\vec{x}$"

time dependence: from Schrödinger to Heisenberg

$$\underset{(t,\vec{x})}{\mathcal{O}_H} = e^{iHt}\,\underset{(\vec{x})}{\mathcal{O}_S}\,e^{-iHt}$$

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_{\vec{p}}\,e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}}\,e^{-i\vec{p}\vec{x}}\right)$$

$$\Rightarrow \phi(x) = \phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\Big\{\underbrace{e^{iHt}a_{\vec{p}}e^{-iHt}}_{a_{\vec{p}}e^{-iE_{\vec{p}}t}}e^{i\vec{p}x} + \underbrace{e^{iHt}a^\dagger_{\vec{p}}e^{-iHt}}_{a^\dagger_{\vec{p}}e^{+iE_{\vec{p}}t}}e^{-i\vec{p}\vec{x}}\Big\}$$

$$\left[\, e^{iHt}a_{\vec{p}} = \sum_n (iHt)^n a_{\vec{p}} \quad \middle| \; [H, a_{\vec{p}}] = -\omega_p a_{\vec{p}}\right.$$

$$= \sum_n a_{\vec{p}}(it(H-\omega_p))^n$$

$$\left. = a_{\vec{p}}\,e^{it(H-\omega_p)} \right]$$

$$\Rightarrow \boxed{\phi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left\{a_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{+ip\cdot x}\right\}_{p^0 = \omega_p > 0}}$$

"positive frequency mode" — "negative frequency mode"

NB: • **inherent duality**: $a, a^\dagger$ – particle interpretation (= quanta of field excitation)

$e^{\pm ipx}$ – wave interpretation

⇝ solutions of KG eq.

- 2 solutions for relativistic wave equation:
  - coefficient of pos. frequency mode destroys a particle w. positive energy
  - " " " neg. " " creates " " " positive energy

particle propagation

amplitude for a particle from y to x:

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^{+} e^{+ip\cdot x} \right\} \left\{ a_{\vec{p}'} e^{-ip'y} + a_{\vec{p}'}^{+} e^{ip'y} \right\} |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} e^{-ipx + ip'y} \langle 0| \underbrace{a_p a_{p'}^{+}}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')} |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

⇝ does not vanish for $(x-y)^2 < 0$, i.e. outside the light cone !? ⌜see PS, p. 250 ff⌟

BUT: only need to require this of observables!

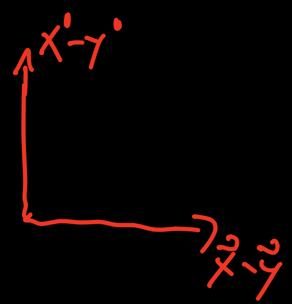
e.g. measurement of $\phi(x)$ and $\phi(y)$

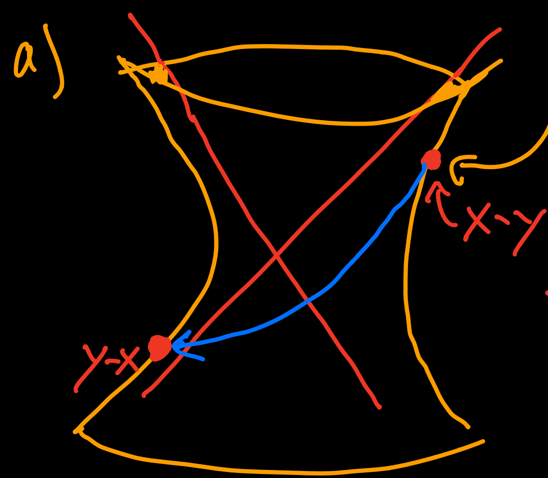
$\rightsquigarrow$ need to consider $[\phi(x), \phi(y)]$!

($=0$ iff the two measurements do not affect each other)

$$[\phi(x), \phi(y)] = \ldots = D(x-y) - D(y-x)$$ [NB: no $\langle 0 | \ldots | 0 \rangle$!]

$$= \begin{cases} 0 & \text{for } (x-y)^2 < 0 \\ \neq 0 & \text{for } (x-y)^2 > 0 \end{cases}$$ ✓



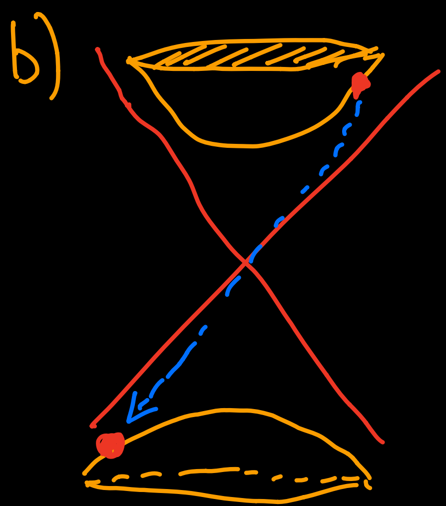


∂V: $(x-y)^2 = \text{const.} < 0$

$\exists$ Lorentz transformation

$x-y \longrightarrow -(x-y)$

$\Rightarrow D(x-y) = D(y-x)$!



$\nexists$ (cont.) Lorentz trafo

$x-y \dashrightarrow -(x-y)$

$\Rightarrow D(x-y) \neq D(y-x)$

Green's functions of Klein-Gordon operator

$$(\partial_x^2 + m^2)\, G(x-y) = -i\,\delta^{(4)}(x-y)$$

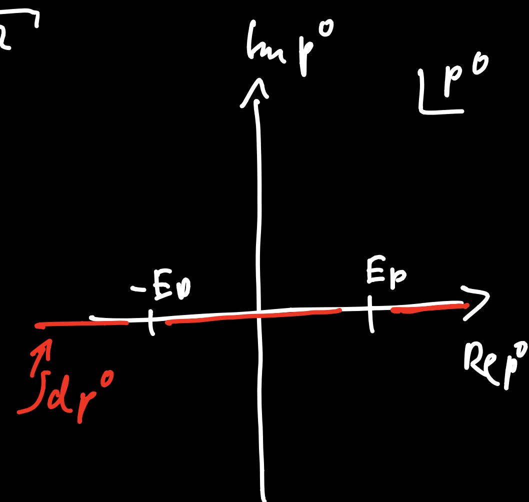
$$\int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, G(p) \qquad\qquad \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}$$

$$(-p^2+m^2)\, G(p) = -i$$

$$\Rightarrow \quad G(p) = \frac{i}{p^2-m^2}$$

$$\Rightarrow \quad \boxed{G(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m^2}\, e^{-ip(x-y)}}$$

poles at $p^0 = \pm E_p = \pm\sqrt{\vec{p}^{\,2}+m^2}$

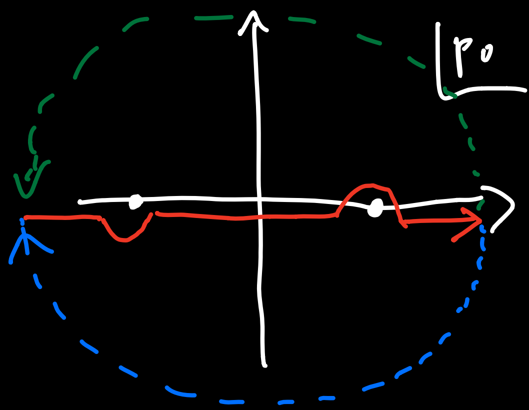


→ 4 ways of treating the poles, i.e. 4 different Green's functions

a) Feynman propagator

$$D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)}$$

$$\hookrightarrow (p^0-E_p)(p^0+E_p)+i\epsilon \Rightarrow p^0 = \pm(E_p - i\epsilon)$$



$\Rightarrow$ i) $x^0 > y^0$ $\Rightarrow$ contours can be closed below

$\Gamma e^{-ip^0(x^0-y^0)} \longrightarrow 0$ for $p^0 \to -i\infty$

$\Rightarrow$ pick up pole at $p^0 = +E_p$

$$\Rightarrow D_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi} \frac{i}{p^0 - E_p} \frac{e^{-ip(x-y)}}{p^0 + E_p}$$

$\oint f(z)\,dz = 2\pi i \, \mathrm{Res}\, f(z_0)$

$p^0 = E_p$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = D(x-y) \;!$$

ii) $x^0 < y^0$: close the contour above

$\Rightarrow e^{-ip^0(x^0-y^0)} \to 0$ for $p^0 \to +i\infty$

$\Rightarrow$ pick up pole at $p^0 = -E_p$

$$\Rightarrow D_F(x-y) = \int \frac{d^3p}{(2\pi)} \oint \frac{dp^0}{2\pi i} \frac{-1}{p^0 + E_p} \frac{e^{-ip\cdot(x-y)}}{p^0 - E_p}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{\boxed{+}ip^0(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$-$ after $\vec{p} \to -\vec{p}$

$$= D(y-x)$$

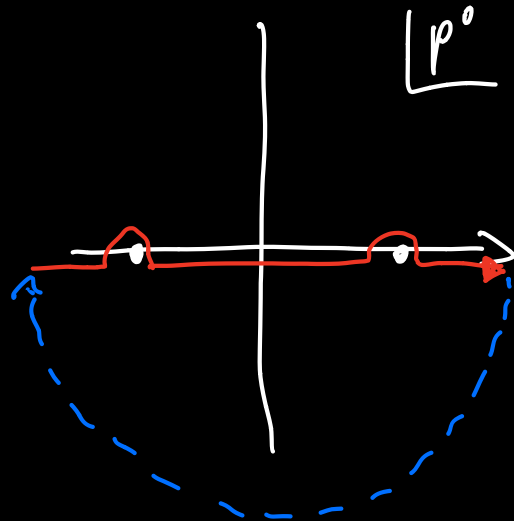
$$\Rightarrow \boxed{D_F(x-y) = \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } y^0 > x^0 \end{cases} \equiv \langle 0 | T \phi(x) \phi(y) | 0 \rangle}$$

"time-ordering" T : order all operators by following the "T", latest to the left.

b) <u>retarded</u> <u>Green's</u> <u>function</u>

(vanishes for $x^0 < y^0$)

$\rightsquigarrow$ take a contour <u>above</u> both poles (need $x^0 > y^0$ to pick up both)



$$\Rightarrow D_R(x-y) = \int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi i} \frac{1}{p^0 - E_p} \frac{1}{p^0 + E_p} e^{-ip\cdot(x-y)}$$

$$= \Theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip(x-y)} - \frac{1}{2E_p} e^{+ip(x-y)} \right\}$$

$$= \Theta(x^0 - y^0) \left\{ D(x-y) - D(y-x) \right\}$$

$$\Rightarrow \boxed{D_R(x-y) = \Theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle}$$

3. The Dirac algebra

recall Lorentz algebra:

$$(*)\quad [J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)$$

- goal: look for a finite-dimensional representation that corresponds to spin $1/2$

⇝ "idea": take $n\times n$ matrices γ^μ with

$$\boxed{\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbb{1}_{n\times n}} \quad (**)$$

"Dirac/Clifford algebra"

$\Rightarrow (\gamma^0)^2 = \mathbb{1}$

$(\gamma^i)^2 = -\mathbb{1}$

$$\Rightarrow \boxed{S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]} \text{ satisfies } (*)\,!$$

⇝ exercise: show this! (warning: rather technical...)

remark: you already "know" this in 3D!

Def. $\gamma^i = i\sigma^i$ Pauli matrices: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \{\gamma^i, \gamma^j\} = -\{\sigma^i, \sigma^j\} = -2\delta^{ij} \checkmark$$

[as required by (**)]

- $S^{ij} = -\frac{i}{4}[\sigma^i, \sigma^j] = \frac{1}{2}\varepsilon^{ijk}\sigma^k$ [c.f. earlier 3D discussion of (*)!]

$\Rightarrow$ Pauli matrices are a representation of the rotation group!

"the spin $\frac{1}{2}$" representation

Lorentz transformation properties

$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$ is called a Dirac spinor if it transforms under Lorentz transformations with $S^{\mu\nu}$, i.e.

$$\boxed{\psi_a(x) \longrightarrow \psi'_a(x) = \underbrace{M(\Lambda)}_{\equiv (\Lambda_{1/2})_{ab}} \psi_b(\Lambda^{-1}x)}$$

with
$$\boxed{\begin{aligned} &\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\tilde{J}^{\mu\nu}\right) \qquad i(\tilde{J}^{\mu\nu})_{ab} = i(\delta^\mu_a \delta^\nu_b - \delta^\mu_b \delta^\nu_a) \\ &\Lambda_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \end{aligned}}$$

- how does the γ^μ "transform"? (NB: γ^μ are constants!)

$\rightsquigarrow$ what we mean is the following:

Consider $\gamma^\mu \psi \longrightarrow \gamma^\mu \Lambda_{1/2} \psi \equiv \Lambda_{1/2} \gamma^{\mu'} \psi$

$$\rightsquigarrow \gamma^{\mu'} = \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2}$$

[for $\omega \ll 1$: $\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \left(\mathbb{1} + \frac{i}{2} \omega_{\rho\sigma} S^{\rho\sigma}\right) \gamma^\mu \left(\mathbb{1} - \frac{i}{2} \omega_{\rho'\sigma'} S^{\rho'\sigma'}\right)$

$$= \gamma^\mu + \frac{i}{2} \omega_{\rho\sigma} \left[S^{\rho\sigma}, \gamma^\mu\right]$$

$$= \gamma^\mu - \frac{1}{8} \omega_{\rho\sigma} \underbrace{\left\{ \underbrace{(\gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho)}_{2(\gamma^\rho\gamma^\sigma - g^{\rho\sigma})} \gamma^\mu - \gamma^\mu \underbrace{(\gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho)}_{2(\gamma^\rho\gamma^\sigma - g^{\rho\sigma})} \right\}}_{\substack{2(\gamma^\rho \gamma^\sigma \gamma^\mu - \gamma^\mu \gamma^\rho \gamma^\sigma) \\ = 4(g^{\sigma\mu} \gamma^\rho - g^{\mu\rho} \gamma^\sigma)}}$$

$$= \gamma^\mu - \frac{1}{2} \omega_{\rho\sigma} \underbrace{\left(g^{\mu\sigma} \delta^\rho_\nu - g^{\mu\rho} \delta^\sigma_\nu\right)}_{\substack{g^{\mu\tau}(\delta^\sigma_\tau \delta^\rho_\nu - \delta^\rho_\tau \delta^\sigma_\nu) \\ = i g^{\mu\tau} (\mathcal{J}^{\rho\sigma})_{\tau\nu}}} \gamma^\nu$$

$$= \left(\mathbb{1} - \frac{i}{2} \omega_{\rho\sigma} \mathcal{J}^{\rho\sigma}\right)^\mu{}_\nu \gamma^\nu$$

$$\Rightarrow \gamma^{\mu'} = \boxed{\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu}$$

i.e. $\gamma^\mu \psi$ transforms like a four vector! + spinor

Some basic facts about γ matrices

a) $\boxed{(\gamma^\mu)^\dagger = (\gamma^\mu)^{-1}}$: can be chosen unitary because they form a rep. of a finite group

⌜consider any rep. of G and a hermitian product $(,)$.

$$\to (x,y)' \equiv \sum_{g\in G} (gx, gy)$$

$$\Rightarrow \forall h \in G : (hx, y)' = \sum_g (ghx, gy)$$

$$= \sum_g (ghx, ghh^{-1}y)$$

"g'"

$$= \sum_{g'} (g'x, g'h^{-1}y)$$

$$= (x, h^{-1}y)' \quad \square$$ ⌟

b) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow$ • $(\gamma^0)^2 = 1 \; (\times \mathbb{1}_{4\times 4}) \quad | \; \mathbb{1}^\dagger = \mathbb{1}$

$$\Rightarrow 1 = (\gamma^{0\dagger})^2 = (\gamma^0)^\dagger (\gamma^0)^{-1}$$

$$\Rightarrow \boxed{(\gamma^0)^\dagger = \gamma^0}$$

• $(\gamma^i)^2 = -1 \Rightarrow \ldots \Rightarrow \boxed{(\gamma^i)^\dagger = -\gamma^i}$

c) $\gamma^{\mu\dagger}\gamma^0 = \left\{\begin{array}{ll} \gamma^0\gamma^0 & \text{for } \mu=0 \\ -\gamma^i\gamma^0 & \text{for } \mu=i \end{array}\right\} = \boxed{\gamma^0\gamma^\mu = \gamma^{\mu\dagger}\gamma^0}$

Dirac bilinears

$\rightsquigarrow$ How to get a Lorentz scalar from ψ?

NB: generators not hermitian, i.e. $(S^{\mu\nu})^\dagger \neq S^{\mu\nu}$

$\Rightarrow \Lambda_{1/2}$ not unitary, i.e. $\Lambda_{\frac{1}{2}}^\dagger \neq \Lambda_{\frac{1}{2}}^{-1}$

$\Rightarrow \psi_a^\dagger \psi_a \xrightarrow{\text{L.T.}} \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \Lambda_{\frac{1}{2}} \psi \neq \psi^\dagger\psi$ ↯

solution: $\boxed{\bar\psi \equiv \psi^\dagger\gamma^0}$

now: $\bar\psi \longrightarrow (\Lambda_{\frac{1}{2}}\psi)^\dagger\gamma^0 \overset{\omega\ll 1}{=} \psi^\dagger(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger})\gamma^0$

$$\left| \begin{array}{l} S^{\mu\nu\dagger} = -\frac{i}{4}[\gamma^\mu,\gamma^\nu]^\dagger \\ = \frac{i}{4}[\gamma^{\mu\dagger},\gamma^{\nu\dagger}] \\ \Rightarrow S^{\mu\nu\dagger}\gamma^0 = \frac{i}{4}\gamma^0[\gamma^\mu,\gamma^\nu] \end{array}\right.$$

$= \psi^\dagger\gamma^0(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu})$

i.e. $\boxed{\bar\psi \to \bar\psi\Lambda_{1/2}^{-1}}$

$\Rightarrow$ • $\bar{\psi}\psi$ transforms like a scalar !

• $\bar{\psi}\gamma^\mu\psi$ " " " vector !

$$\left[\ \bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}_a \underbrace{\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}}_{\Lambda^\mu{}_\nu \gamma^\nu_{ab}}\psi_b = \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi\ \right]$$

• $\bar{\psi}S^{\mu\nu}\psi$ " " " tensor !

lowest possible n in 4D: $n=4$

⇝ we will consider 4×4 γ matrices here

Q: How to decompose a general $\Gamma = 4\times4$ matrix into basis elements Γ_i such that $\bar{\psi}\Gamma_i\psi$ has definite transformation properties under Lorentz transformations?

⇝ need to introduce one more

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}$$

$\Rightarrow$
- $(\gamma^5)^\dagger = \gamma^5$
- $(\gamma^5)^2 = \mathbb{1}$
- $\{\gamma^5, \gamma^\mu\} = 0$

basis elements of 4×4 matrices (Γ_i)	#	LT properties ($\bar{\psi}\Gamma_i\psi$)
$\mathbb{1}$	1	scalar
γ^μ	4	vector
$\sigma^{\mu\nu} \equiv 2S^{\mu\nu}$	6	tensor

γ^5	1	pseudo-scalar
$\gamma^\mu\gamma^5$	4	pseudo-vector / axial vector
	16 ✓	

⌜ • first determine $\Lambda_{1/2}$ for $\boxed{\text{reflections}}$, i.e. $\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$\Rightarrow \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu = \begin{cases} \gamma^0 & (\mu=0) \\ -\gamma^i & (\mu=i) \end{cases}$$

solution : $\boxed{\Lambda_{1/2} = \eta_p \gamma^0}$ $\Rightarrow \Lambda_{1/2}^{-1} = \eta_p^* \gamma^0$

phase, i.e. $|\eta_p| = 1$

$$\Rightarrow \bullet\ \underset{(t,\vec{x})}{\bar\psi}\gamma^5 \underset{(t,\vec{x})}{\psi} \xrightarrow{\vec{x}\to-\vec{x}} \bar\psi \Lambda_{1/2}^{-1}\gamma^5\Lambda_{1/2}\psi = \bar\psi\gamma^0\underbrace{\gamma^5\gamma^0}_{-\gamma^0\gamma^5}\psi = -\bar\psi\gamma^5\psi\ ✓$$

$$\bullet\ \bar\psi\gamma^\mu\gamma^5\psi \xrightarrow{\vec{x}\to-\vec{x}} \bar\psi\gamma^0\gamma^\mu\gamma^5\gamma^0\psi = \bar\psi\begin{Bmatrix} \gamma^0 & (\mu=0) \\ -\gamma^i & (\mu=i) \end{Bmatrix}\underbrace{\gamma^0\gamma^5\gamma^0}_{-\gamma^5}\psi$$

$$= -\bar\psi\gamma^\mu\gamma^5\psi \begin{cases} +1 & \text{for } \mu=0 \\ -1 & \text{for } \mu=i \end{cases}$$

✓ ⌟

representations of Dirac matrices

Lowest possible n in 4D : $n=4$

"Weyl" or "chiral" rep :

Pauli matrices

$$\boxed{\gamma^0 = \begin{pmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0_{2\times2} \end{pmatrix} \; ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

$$\Rightarrow \boxed{\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}}$$

$\Rightarrow$ boosts : $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = -(S^{0i})^\dagger$

rotations : $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

$$\equiv \frac{1}{2} \varepsilon^{ijk} \Sigma^k = (S^{ij})^\dagger$$

4. The Dirac equation

goal : find a relativistic wave equation for Dirac spinors

⇝ there exists a 1st order Lorentz-invariant equation!

$$\boxed{\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0}} \qquad \text{"Dirac equation"} \quad (*)$$

$$\begin{aligned}
\Rightarrow \quad 0 &= (-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)\,\psi \\
&= (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\,\psi \qquad \big|\ \partial_\nu \partial_\mu = \partial_\mu \partial_\nu \\
&= \Big(\underbrace{\tfrac{1}{2}\{\gamma^\nu, \gamma^\mu\}}_{g^{\mu\nu}} \partial_\nu \partial_\mu + m^2\Big)\,\psi \\
&= (\partial^2 + m^2)\,\psi
\end{aligned}$$

⇒ every spinor field satisfying $(*)$ also satisfies KG eq., i.e. correct p^μ–m relation!

$$\Leftrightarrow \quad \boxed{\boxed{\mathcal{L} = \bar\psi\,(i\not{\partial} - m)\,\psi}} \qquad \text{where } \not{A} \equiv \gamma^\mu A_\mu$$

Weyl spinors

recall block-diagonal form of $S^{\mu\nu}$

⇝ Dirac rep. of the Lorentz group is reducible !

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$: left- / righthanded "Weyl spinors"

(2 components ↔ 4 = of Dirac spinors)

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\Rightarrow \boxed{\underbrace{\psi_L}_{= \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}} = \underbrace{\left(\frac{1-\gamma^5}{2}\right)}_{\equiv P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \psi \,, \quad \underbrace{\psi_R}_{= \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}} = \underbrace{\frac{1+\gamma^5}{2}}_{\equiv P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \psi}$$

$\Rightarrow$ Dirac eq.: $0 = (i\gamma^\mu \partial_\mu - m)\psi$

$$= \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \\ i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$\sigma^\mu = (1, \vec{\sigma})$

$\tilde{\sigma}^\mu = (1, -\vec{\sigma})$

$$= \begin{pmatrix} -m & i\sigma \cdot \partial \\ i\tilde{\sigma} \cdot \partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$\Rightarrow$ • $m \neq 0$ mixes ψ_R, ψ_L

• $m = 0$: $\boxed{\begin{aligned} \tilde{\sigma} \cdot \partial \psi_L &= 0 \\ \sigma \cdot \partial \psi_R &= 0 \end{aligned}}$ "Weyl equations"

$\rightsquigarrow$ neutrinos...

Conserved currents

• "vector current" $j^\mu \equiv \bar{\psi} \gamma^\mu \psi$

$$\Rightarrow \partial_\mu j^\mu = \underbrace{(\partial_\mu \bar{\psi}) \gamma^\mu}_{-im\bar{\psi}} \psi + \bar{\psi} \underbrace{\gamma^\mu \partial_\mu \psi}_{im\psi}$$

| Dirac eq: $(i\partial\!\!\!/ + m)\psi = 0$

$\Rightarrow \psi^\dagger(-i\overleftarrow{\partial\!\!\!/}^\dagger + m) = 0$

$\Rightarrow \psi^\dagger(-i\overleftarrow{\partial\!\!\!/}^\dagger + m)\gamma^0 = 0$

$\Rightarrow \underbrace{\psi^\dagger \gamma^0}_{\bar{\psi}}(-i\overleftarrow{\partial\!\!\!/} + m) = 0$

$$\Rightarrow \boxed{\partial_\mu j^\mu = 0}$$

• "axial vector current"

$$j^{\mu 5} \equiv \bar{\psi} \gamma^\mu \gamma^5 \psi$$

$$\Rightarrow \boxed{\partial_\mu j^{\mu 5} = \ldots = 2im\bar{\psi}\gamma^5\psi}$$

$\rightsquigarrow$ conserved if $m = 0$!

similar: $j_L^\mu \equiv \bar{\psi}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi = \bar{\psi}_L\gamma^\mu\psi_L$

$$j_R^\mu \equiv \bar{\psi}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi = \bar{\psi}_R\gamma^\mu\psi_R$$

$\rightsquigarrow$ ψ_L and ψ_R can have different charges!

free-particle solutions

ψ obeys KG eq.

$$\Rightarrow \boxed{\psi(x) = u(p)\, e^{-ipx} + v(p)\, e^{+ipx}} \quad \text{with } p^2 = m^2,\ p^0 > 0$$

4-component spinors, independent of x

a) positive frequency : determine $u(p)$

strategy : i) use Dirac eq. for $p = p_{rest} = (m, \vec{0})$

ii) then boost with $\Lambda_{1/2}$ to arbitrary p^μ

i) $(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (\underbrace{p\!\!\!/}_{p_\mu \gamma^\mu} - m)\, u(p) = 0 \quad | \, p = p_{rest}$

$$\Rightarrow (m\gamma^0 - m)\, u(p_{rest}) = 0$$

$$\Leftrightarrow m \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} u(p_{rest}) = 0$$

$$\Rightarrow \boxed{u(p_{rest}) \propto \begin{pmatrix} \xi \\ \xi \end{pmatrix}}$$

ξ : arbitrary 2 - component spinor

$\Rightarrow$ two independent possibilities: $\xi^r \, ; r = 1,2$

with $\boxed{\xi^{r\dagger} \xi^s = \delta^{rs}}$

$(\rightsquigarrow \xi = a \cdot \xi^1 + b \, \xi^2)$

interpretation: recall rotation generator:

$$S^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

block diagonal $\Rightarrow$ ξ transforms exactly like a 2-component spinor (spin $\frac{1}{2}$) in QM!

e.g. $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: spin up along z-direction

ii) now boost

- boost of a 4-vector:

$$\begin{pmatrix} E \\ \vec{p} \end{pmatrix} = p^\mu = \Lambda^\mu{}_\nu \, p^\nu_{rest} = \Lambda^\mu{}_\nu \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$\Lambda = \exp\left[-\frac{i}{2} w_{g\sigma} (\mathcal{J}^{g\sigma})^{\cancel{\mu}}{}_{\cancel{\nu}}\right]^\mu{}_\nu$$

exercises: $\omega_{10} = -\omega_{01} \equiv \eta$

"rapidity"

(describes boost in x-direction)

general boost:

$$\omega_{\rho\sigma} = \eta \left(\begin{array}{c|c} 0 & \hat{p}^T \\ \hline -\hat{p} & 0_{3\times 3} \end{array}\right)$$

$$\hat{p} \equiv \frac{\vec{p}}{|\vec{p}|}$$

$$\Rightarrow \Lambda^\mu{}_\nu = \exp\left[-i\,\omega_{0i} (\tilde{J}^{0i})^\mu{}_\nu\right]$$

$$(\tilde{J}^{0i})_{\mu\nu} = i\left(\delta^0_\mu \delta^i_\nu - \delta^i_\mu \delta^0_\nu\right)$$

$$= i \left(\begin{array}{c|c} 0 & \hat{e}_i^T \\ \hline -\hat{e}_i & 0_{3\times 3} \end{array}\right)_{\mu\nu} \qquad \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ldots$$

$$\Rightarrow (\tilde{J}^{0i})^\mu{}_\nu = i \begin{pmatrix} 0 & \hat{e}_i^T \\ +\hat{e}_i & 0 \end{pmatrix}^\mu{}_\nu$$

$$= \exp\left[\eta \begin{pmatrix} 0 & \hat{p}^T \\ \hat{p} & 0_{3\times 3} \end{pmatrix}\right]$$

$$= \mathbb{1} + \eta \begin{pmatrix} 0 & \hat{p}^T \\ \hat{p} & 0_{3\times 3} \end{pmatrix} + \frac{\eta^2}{2} \begin{pmatrix} 1 & 0 \\ 0 & \hat{p}\hat{p}^T \end{pmatrix}$$

$$+ \frac{\eta^3}{3!} \begin{pmatrix} 0 & \hat{p}^T \\ \hat{p} & 0 \end{pmatrix} + \ldots$$

$$\Rightarrow \Lambda^\mu{}_\nu \begin{pmatrix} m \\ 0 \end{pmatrix}^\nu = m \Lambda^\mu{}_0$$

$$\Rightarrow \boxed{\begin{aligned} E &= m\Lambda^0{}_0 = m \cdot \cosh\eta \\ p^i &= m\Lambda^i{}_0 = m \cdot \hat{p}^i \sinh\eta \end{aligned}}$$

- now boost spinor correspondingly:

$$u(p) = \Lambda_{1/2}\, u(p_{rest})$$

where $\Lambda_{1/2} = \exp[-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}]$ $\quad \Big|\ S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

$$= \exp[-i\omega_{0i} S^{0i}]$$

$$= \exp\left[-\frac{\eta}{2}\begin{pmatrix} \hat{p}\cdot\vec{\sigma} & 0 \\ 0 & -\hat{p}\cdot\vec{\sigma} \end{pmatrix}\right]$$

$$= \mathbb{1} - \frac{\eta}{2}\begin{pmatrix} \hat{p}\cdot\vec{\sigma} & 0 \\ 0 & -\hat{p}\cdot\vec{\sigma} \end{pmatrix} + \frac{1}{2}\left(\frac{\eta}{2}\right)^2 \begin{pmatrix} (\hat{p}\cdot\vec{\sigma})^2 & 0 \\ 0 & (\hat{p}\cdot\vec{\sigma})^2 \end{pmatrix}$$

$\to \hat{p}_i \hat{p}_j \underbrace{\sigma_i \sigma_j}_{\to \frac{1}{2}\{\sigma_i, \sigma_j\}}$

$= \hat{p}_i \hat{p}_j \{\delta_{ij} + \cancel{\epsilon_{ijk}\sigma_k}\}$

$= 1$

$\Rightarrow (\quad) = \mathbb{1}\ !$

$$-\frac{1}{3!}\left(\frac{\eta}{2}\right)^3 \begin{pmatrix} \hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}$$

$$\Rightarrow \Lambda_{1/2} = \cosh\frac{\eta}{2}\cdot\mathbb{1} - \sinh\frac{\eta}{2}\begin{pmatrix}\hat{p}\cdot\vec{\sigma} & 0\\ 0 & -\hat{p}\cdot\vec{\sigma}\end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{\eta}{2}}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right)+e^{-\frac{\eta}{2}}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right) & 0\\ 0 & e^{\frac{\eta}{2}}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right)+e^{-\frac{\eta}{2}}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right)\end{pmatrix}$$

$$e^{\pm\frac{\eta}{2}} = \cosh\frac{\eta}{2} \pm \sinh\frac{\eta}{2}$$
$$= \sqrt{\cosh\eta \pm \sinh\eta}$$
$$= \sqrt{E/m \pm |\vec{p}|/m}$$

$$= \frac{1}{2\sqrt{m}}\begin{pmatrix}\sqrt{E+|\vec{p}|}+\sqrt{E-|\vec{p}|} - \hat{p}\cdot\vec{\sigma}\left(\sqrt{E+|\vec{p}|}-\sqrt{E-|\vec{p}|}\right) & 0\\ 0 & {}'' \; + \; {}''\end{pmatrix}$$

$$\sqrt{E+|\vec{p}|} \pm \sqrt{E-|\vec{p}|} = \sqrt{(E+|\vec{p}|)+(E-|\vec{p}|)\pm 2\underbrace{\sqrt{E^2-\vec{p}^2}}_{m}}$$
$$= \sqrt{2(E\pm m)}$$

$$= \frac{1}{\sqrt{2m}}\begin{pmatrix}\sqrt{E+m}-\sqrt{E-m}\,\hat{p}\cdot\vec{\sigma} & 0\\ 0 & \sqrt{E+m}+\sqrt{E-m}\,\hat{p}\cdot\vec{\sigma}\end{pmatrix}$$

$$|\vec{p}| = \sqrt{E^2-m^2} = \sqrt{E+m}\,\sqrt{E-m}$$

$$\Rightarrow \sqrt{E-m}\,\hat{p} = \frac{\vec{p}}{\sqrt{E+m}}$$

$$= \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m-\vec{p}\cdot\vec{\sigma} & 0 \\ 0 & E+m+\vec{p}\cdot\vec{\sigma} \end{pmatrix}$$

$$\Rightarrow u(p) = \Lambda_{\frac{1}{2}}\, u(p_{rest})$$

$$\Rightarrow \boxed{u^{r} = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} [E+m-\vec{p}\cdot\vec{\sigma}]\,\xi^{r} \\ [E+m+\vec{p}\cdot\vec{\sigma}]\,\xi^{r} \end{pmatrix}}$$

$r=1,2$

normalization convention

NB: two independent solutions! ("spin up and down")

<u>example</u>: $\vec{p} = (0,0,p) \Rightarrow \vec{p}\cdot\vec{\sigma} = p\sigma^{3} = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$

- $E+m \pm p = \sqrt{\underbrace{(E+m)^{2}+p^{2}}_{E^2-m^2\text{ in }p^2} \pm 2p(E+m)}$

$$= \sqrt{2}\,\sqrt{E^{2}+Em \pm p(E+m)}$$

$$= \sqrt{2}\,\sqrt{E+m}\,\sqrt{E\pm p}$$

$$\Rightarrow u^r(p) = \begin{pmatrix} \sqrt{E-p} & 0 & 0 & 0 \\ 0 & \sqrt{E+p} & 0 & 0 \\ 0 & 0 & \sqrt{E+p} & 0 \\ 0 & 0 & 0 & \sqrt{E-p} \end{pmatrix} \begin{pmatrix} \xi^r \\ \xi^r \end{pmatrix}$$

- normalization:

$$u^{r\dagger} u^s = \frac{1}{2(E+m)} \left(\xi^{r\dagger} [E+m - \vec{p}\vec{\sigma}],\ \xi^{r\dagger} [E+m+\vec{p}\vec{\sigma}] \right) \times$$

$\sigma^{i\dagger} = \sigma^i$

$$\times \begin{pmatrix} [E+m - \vec{p}\vec{\sigma}]\,\xi^s \\ [E+m+\vec{p}\vec{\sigma}]\,\xi^s \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \xi^{r\dagger} \underbrace{\left(\underbrace{[E+m-\vec{p}\vec{\sigma}]^2}_{2\times 2 \text{ matrix!}} + [E+m+\vec{p}\vec{\sigma}]^2 \right)}_{2(E+m)^2 \times \mathbb{1}_{2\times 2} + 2(\vec{p}\vec{\sigma})^2} \xi^s$$

$p_i p_j \sigma_i \sigma_j = \vec{p}^{\,2} \cdot \mathbb{1}_{2\times 2}$

$$= \frac{(E+m)^2 + \vec{p}^{\,2}}{E+m} \underbrace{\xi^{r\dagger} \xi^s}_{\delta^{rs}}$$

$\big| \ \vec{p}^{\,2} = (E+m)(E-m)$

$$= \underline{\underline{2E\,\delta^{rs}}}$$

similar: $\boxed{\bar{u}^r u^s = \ldots = 2m\delta^{rs}}$

b) negative frequency solutions

$$\psi(x) = v(p)\, e^{+ipx} \quad ; \; p^2 = m^2, \; p^0 > 0$$

$\rightsquigarrow \ldots$

$v^s(p_{rest}) \propto \begin{pmatrix} \eta^s \\ \eta^s \end{pmatrix}$

$$\boxed{\boxed{v^s = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} [E+m-\vec{p}\vec{\sigma}]\,\eta^s \\ -[E+m+\vec{p}\vec{\sigma}]\,\eta^s \end{pmatrix}}}$$

with $\eta^{r\dagger}\eta^s = \delta^{rs}$ $\quad (r,s = 1,2)$

$\rightsquigarrow$ also 2 solutions, in total 4

$\Rightarrow (\ldots)$ $\boxed{\begin{aligned} v^{r\dagger} v^s &= +2E\delta^{rs} \\ \bar{v}^r v^s &= -2m\delta^{rs} \end{aligned}}$

also $(\ldots)$

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$$

$$u^{r\dagger}(p)\, v^s(-p) = v^{r\dagger}(p)\, u^s(-p) = 0$$

Spin sums

$$\boxed{\begin{aligned}&\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \not{p} + m\\[1em]&\sum_{s=1,2} v^s(p)\,\bar{v}^s(p) = \not{p} - m\end{aligned}}$$

$$\Gamma \sum_{s=1,2} u^s(p)\, u^{s\dagger}(p)\, \gamma^0$$

$$= \frac{1}{2(E+m)} \sum_{s=1,2} \begin{pmatrix} [E+m-\vec{p}\vec{\sigma}]\,\xi^s \\ [E+m+\vec{p}\vec{\sigma}]\,\xi^s \end{pmatrix} \times \left(\xi^{s\dagger}[E+m-\vec{p}\vec{\sigma}],\ \xi^{s\dagger}[E+m+\vec{p}\vec{\sigma}] \right)$$

$$\times \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$= \frac{1}{2(E+m)} \sum_{s=1,2} \begin{pmatrix} [E+m-\vec{p}\vec{\sigma}]\,\xi^s \\ [E+m+\vec{p}\vec{\sigma}]\,\xi^s \end{pmatrix} \times \left(\xi^{s\dagger}[E+m+\vec{p}\vec{\sigma}],\ \xi^{s\dagger}[E+m-\vec{p}\vec{\sigma}] \right)$$

$$\left|\ \sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right.$$

$$= \frac{1}{2(E+m)} \begin{pmatrix} (E+m)^2 - \underbrace{(\vec{p}\vec{\sigma})^2}_{\vec{p}^2} & (E+m)^2 - 2\vec{p}\vec{\sigma}(E+m) + \vec{p}^2 \\ (E+m)^2 + 2\vec{p}\vec{\sigma}(E+m) + \vec{p}^2 & (E+m)^2 - \vec{p}^2 \end{pmatrix}$$

NB: every entry is a 2x2 matrix!

$$\left| \; \vec{p}^{\,2} = (E+m)(E-m) \right.$$

$$= \frac{1}{2} \begin{pmatrix} E+m-(E-m) & E+m+(E-m)-2\vec{p}\vec{\sigma} \\ E+m+(E-m)+2\vec{p}\vec{\sigma} & E+m-(E-m) \end{pmatrix}$$

$$= \begin{pmatrix} m & E-\vec{p}\sigma \\ E+\vec{p}\sigma & m \end{pmatrix}$$

$$= m \cdot \mathbb{1}_{4\times 4} + p_\mu \gamma^\mu \quad \square \quad \lrcorner$$

5. Quantizing the Dirac field

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \psi^+(i\partial_t + i\gamma^0\vec{\gamma}\vec{\nabla} - m)\psi$$

$$\Rightarrow \pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^+$$

$$\Rightarrow H = \int d^3x\,(\pi\dot{\psi} - \mathcal{L})$$

$$= \int d^3x\,\psi^+\underbrace{(-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0)}_{\equiv h_D}\psi$$

energy eigenvalues

goal: diagonalize H like for scalar field

$\rightarrow$ need to identify all 4 eigenfunctions of $(\gamma)h_D$!

recall Dirac equation:

$$[i\cancel{\gamma^0}\partial_t + \underbrace{i\vec{\gamma}\vec{\nabla} - m}_{-\cancel{\gamma_0}h_D}]u^s(\vec{p})e^{-ipx} = 0 \qquad \Big|\; i\partial_t e^{-ipx} = p^0 e^{-ip^0t + i\vec{p}\vec{x}}$$

$$\Rightarrow h_D u^s(\vec{p})e^{+i\vec{p}\vec{x}} = +p^0 u^s(\vec{p})e^{+i\vec{p}\vec{x}}$$

$$\cdot\ h_D\, v^s(\underset{\downarrow\, -\vec p}{\vec p})\, e^{\underset{\downarrow\, +}{-}i\vec p\vec x} = -p^0\, v^s(\underset{\downarrow\, -\vec p}{\vec p})\, e^{\underset{\downarrow\, +}{-}i\vec p\vec x}$$

expand in this basis, promote ψ to operator:

$$\boxed{\psi_a(\vec x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\, e^{+i\vec p\vec x} \sum_{s=1,2}\left(a^s_{\vec p}\, u^s_a(\vec p) + \tilde b^s_{-\vec p}\, v^s_a(-\vec p)\right)}$$

Schrödinger picture
$\rightsquigarrow$ no t-dependence

$$\Rightarrow H = \int d^3x\, \psi^\dagger h_D \psi$$

$$= \int d^3x \int \frac{d^3p'\, d^3p}{(2\pi)^6}\frac{1}{2\sqrt{E_p E_{p'}}}\, e^{-i\vec p\vec x} \sum_{r,s=1,2}\left(a^{s\dagger}_{\vec p}\, u^{s\dagger}(\vec p) + \tilde b^{s\dagger}_{-\vec p}\, v^{s\dagger}(-\vec p)\right)$$

$$\times\ \underbrace{h_D\, e^{+i\vec p'\vec x}\left(a^r_{\vec p'}\, u^r(\vec p') + \tilde b^r_{-\vec p'}\, v^r(-\vec p')\right)}_{E_{p'}\, e^{+i\vec p'\vec x}\left(a^r_{\vec p'}\, u^r(\vec p') - \tilde b^r_{-\vec p'}\, v^r(-\vec p')\right)}$$

$$\Big|\ \int d^3x\, e^{i\vec x(\vec p' - \vec p)} = (2\pi)^3\, \delta^{(3)}(\vec p' - \vec p)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\cancel{E_p}} \sum_{r,s}\left(a^{s\dagger}_{\vec p}\, u^{s\dagger}(\vec p) + \tilde b^{s\dagger}_{-\vec p}\, v^{s\dagger}(-\vec p)\right)$$

$$\times\left(a^r_{\vec p}\, u^r(\vec p) - \tilde b^r_{-\vec p}\, v^r(-\vec p)\right)$$

- $u^{s\dagger} u^{r} = 2E_p\,\delta^{rs}$
- $v^{s\dagger} v^{r} = 2E_p\,\delta^{rs}$
- $u^{s\dagger}(\vec p)\, v^{r}(-\vec p) = v^{s\dagger}(-\vec p)\, u^{r}(\vec p) = 0$

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\, E_p \sum_s \left(a^{s\dagger}_{\vec p}\, a^{s}_{\vec p} - \tilde b^{s\dagger}_{\cancel{-}\vec p}\, \tilde b^{s}_{\cancel{-}\vec p} \right)}$$

problem...

getting rid of negative energies

postulate anti-commutation relations!

$$\boxed{\{a^s_{\vec p}, a^{r\dagger}_{\vec p'}\} \overset{!}{=} \{\tilde b^{s}_{\vec p}, \tilde b^{r\dagger}_{\vec p'}\} \overset{!}{=} (2\pi)^3\, \delta^{(3)}(\vec p - \vec p')\,\delta^{rs}}$$

$+\ \{\ ,\ \} = 0$ otherwise

$\Rightarrow b^s_{\vec p} \equiv \tilde b^{s\dagger}_{\vec p}$ satisfies the same relations!

$\Rightarrow$ a)
$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\, E_p \sum_{s=1,2} \left(a^{s\dagger}_{\vec p}\, a^{s}_{\vec p} + b^{s\dagger}_{\vec p}\, b^{s}_{\vec p} \right)}\ \text{“}-\infty\text{”}$$

↳ const, disregard like in scalar case!

same: $\vec P$ 〃 $\vec p$ 〃 〃 〃

b) $\{\psi(\vec x), i\psi^\dagger(\vec y)\}$

$$= i\int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \sum_{r,s} e^{i\vec{p}\vec{x} - i\vec{p}'\vec{y}} \left\{ \left(a^s_{\vec{p}} u^s(\vec{p}) + b^{s\dagger}_{-\vec{p}} v^s(-\vec{p})\right), \left(a^{r\dagger}_{\vec{p}'} u^{r\dagger}(\vec{p}') + b^r_{-\vec{p}'} v^{r\dagger}(-\vec{p}')\right) \right\}$$

$$\{a^s_{\vec{p}}, a^{r\dagger}_{\vec{p}'}\} u^s(\vec{p}) u^{r\dagger}(\vec{p}') + \{b^{s\dagger}_{-\vec{p}}, b^r_{-\vec{p}'}\} v^s(-\vec{p}) v^{r\dagger}(-\vec{p}')$$

$$= i\int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}(\vec{x}-\vec{y})}}{2E_p} \underbrace{\sum_s \left(u^s(\vec{p})\bar{u}^{s\cancel{*}}(\vec{p}') + v^s(-\vec{p})\bar{v}^{s\cancel{*}}(-\vec{p}) \right)}_{\underbrace{\not{p}}_{\gamma^0 p^0 + \cancel{\gamma^i p^i}} + \cancel{m} + (\gamma^0 p^0 - \cancel{\gamma^i p^i} - \cancel{m})} \gamma^0$$

$$= i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} \frac{1}{\cancel{E_p}} \cancel{p^0} \times \mathbb{1}_{4\times 4}$$

$$\iff \boxed{\{\psi_a(\vec{x}), i\psi_b^\dagger(\vec{y})\} = i\delta(\vec{x}-\vec{y})\,\delta_{ab}}$$

⇝ "expected" (but $[\,,\,] \to \{\,,\,\}$)

Spin → statistics

$$\{ a^{s\,\dagger}_{\vec p}, a^{r\,\dagger}_{\vec k} \} = 0 \quad \Rightarrow \quad \bullet \ (a^{s\,\dagger}_{\vec p})^2 |0\rangle = 0 \quad (*)$$

⇝ only one particle in state $(\vec p, s)$ possible!

$$\bullet \ a^{s\,\dagger}_{\vec p} a^{r\,\dagger}_{\vec k} |0\rangle = - a^{r\,\dagger}_{\vec k} a^{s\,\dagger}_{\vec p} |0\rangle$$

⇒ particles described by Dirac equation (+ anti-commutation relations) obey Fermi-Dirac statistics!

more general <u>theorem</u> by Pauli:

1) Lorentz invariance
2) $E_{\vec p} > 0$
3) positive norms
4) causality

$\Big\}$ ⇒ particles with integer / half-integer spin obey Bose-Einstein / Fermi-Dirac statistics!

<u>remark</u>: for every $(\vec{p}, s)$, there exists only two states <u>defined</u> by $|0\rangle$ and $|1\rangle$

$\Rightarrow$ 2 options: i) $b|0\rangle \equiv 0 \rightsquigarrow b^\dagger|0\rangle \equiv |1\rangle$

ii) $\tilde{b}|0\rangle \equiv 0$, $\tilde{b}^\dagger|0\rangle = |1\rangle$

$(\Leftrightarrow\ b|0\rangle = |1\rangle,\ b^\dagger|0\rangle = 0)$

<u>physical</u> choice: denote the state of lower energy ("vacuum") with $|0\rangle$!

$\Rightarrow$ i) $\langle 0| H = E b^\dagger b |0\rangle = 0$

$< \langle 1| E b^\dagger b |1\rangle$

ii) $\langle 0| H = -E\hat{b}^\dagger\hat{b}|0\rangle = 0 > \langle 1| -E\hat{b}^\dagger\hat{b}|1\rangle$

$= -E$ $\lightning$

full $x=(t,\vec{x})$ dependence

as before: Schrödinger → Heisenberg

i.e. $\psi(x) = e^{iHt}\,\psi(\vec{x})\,e^{-iHt}$

$$\Rightarrow \boxed{\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left(a^s_{\vec{p}}\,u^s(p)\,e^{-ipx} + b^{s\,\dagger}_{\vec{p}}\,v^s(p)\,e^{+ipx}\right)}$$

$\mid p^0 = E_p = \sqrt{\vec{p}^{\,2}+m^2}$

$a^\dagger$ creates "fermions"
$b^\dagger$ creates "anti-fermions"
} both with $E_p > 0$

1-particle states normalized as before:

$$|\vec{p},s\rangle \equiv \sqrt{2E_{\vec{p}}}\;a^{s\,\dagger}_{\vec{p}}\,|0\rangle$$

$$\Rightarrow \langle \vec{p},r\,|\,\vec{q},s\rangle = 2E_{\vec{p}}\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta^{rs}$$

1 antifermion @ x: $\psi(x)\,|0\rangle$

1 fermion @ x: $\bar{\psi}(x)\,|0\rangle$

(electric) charge

recall: $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved

$$\Rightarrow Q = \int d^3x \, j^0 = \int d^3x \, \psi^\dagger \psi$$

$$\vdots$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^s_{\vec{p}} b^{s\dagger}_{\vec{p}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right) \; [+\infty]$$

"charge of vacuum"

$\Rightarrow \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ creates $\begin{pmatrix} \text{fermions} \\ \text{anti-fermions} \end{pmatrix}$ with charge $\begin{pmatrix} +1 \\ -1 \end{pmatrix}$

= const. × electric charge

Dirac propagator

amplitude for fermion to propagate from y to x:

$$\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \sum_{r,s} \langle 0 | \left(a^r_{\vec{p}} u^r_a(p) e^{-ipx} + \cancel{b^{r+}_{\vec{p}} v^r_a e^{ipx}} \right)$$

$$\times \left(a^{s+}_{\vec{p}'} \bar{u}^s_b(\vec{p}') e^{+ip'y} + \cancel{b^s_{\vec{p}'} \bar{v}^s_b(\vec{p}') e^{-ip'y}} \right) | 0 \rangle$$

$$\rightarrow (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \delta^{rs}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_r u^r_a(\vec{p}) \bar{u}^r_b(\vec{p})}_{(\not{p}+m)_{ab}} e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{ab} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}}_{= D(x-y)\ [\equiv \langle 0 | \phi(x) \phi(y) | 0 \rangle]}$$

similar: $\langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle = \ldots = -(i\not{\partial}_x + m)_{ab} D(y-x)$

Green's functions of Dirac equation:

$$(i\not\partial_x - m)\, G(x-y) = i\,\delta^{(4)}(x-y)\cdot \mathbb{1}$$

$$\Leftrightarrow \int \frac{d^4p}{(2\pi)^4}\,(\underbrace{i\not\partial_x}_{\to \not p} - m)\, G(p)\, e^{-ip(x-y)} = \int \frac{d^4p}{(2\pi)^4}\, i\, e^{-ip(x-y)}$$

$$\Rightarrow \quad G(p) = \frac{i}{\not p - m} = \frac{i(\not p + m)}{p^2 - m^2}$$

$\not A \not A = A^2 \cdot \mathbb{1}_{4\times 4}$

$\Rightarrow$ Feynman propagator

$$\boxed{S_F(x-y)_{ab} \equiv \int \frac{d^4p}{(2\pi)^4}\, \frac{i(\not p + m)_{ab}}{p^2 - m^2 + i\varepsilon}\, e^{-ip(x-y)}}$$

$$= (i\not\partial_x + m)_{ab} \underbrace{\int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - m^2 + i\varepsilon}\, e^{-ip(x-y)}}_{D_F(x-y)\, \equiv\, \langle 0|T\phi(x)\phi(y)|0\rangle}$$

$$= \begin{cases} \langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle & \text{for } y^0 > x^0 \end{cases}$$

$$\boxed{S_F(x-y)_{ab} \equiv \langle 0 | T\, \psi_a(x)\, \bar{\psi}_b(x) | 0 \rangle}$$

NB: Definition includes extra minus sign for every field (anti-) commutation necessary to achieve time ordering!

Spin of Dirac fermions

→ use Noether's theorem to derive angular momentum

= conserved charge from invariance under rotations

$$\psi(x) \to \psi'(x) = \Lambda_{1/2}\, \psi(\Lambda^{-1}x) \equiv \psi(x) + \theta \Delta\psi + \mathcal{O}(\theta^2)$$

small rotation by angle θ, around z-axis:

- $\Lambda_{1/2} \simeq \mathbb{1} - \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}$ $\quad | \; \omega_{12} = -\omega_{21} = \theta$

$S^{ij} = \frac{1}{2}\varepsilon^{ijk}\Sigma^k$

$= \mathbb{1} - \frac{i}{2}\theta\, \Sigma^3$ $\quad \Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

- $\Lambda^{-1}x = (t, \cos\theta\, x + \sin\theta\, y, \cos\theta\, y - \sin\theta\, x, z)$

$\simeq (t, x+\theta y, y-\theta x, z) + \mathcal{O}(\theta^2)$

$$\Rightarrow \theta \Delta\psi = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x) - \psi(x)$$

$$= (\mathbb{1} - \tfrac{i}{2}\theta \Sigma^3)\psi(t, x+\theta y, y-\theta x, z) - \psi(x)$$

$$= \theta(y\partial_x - x\partial_y - \tfrac{i}{2}\Sigma^3)\psi(x)$$

$\Rightarrow$ conserved charged density:

$$j^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi = -i\psi^\dagger(x\partial_y - y\partial_x + \tfrac{i}{2}\Sigma^3)\psi$$

$\rightsquigarrow$ analogously for rotations around x, y axis

$$\boxed{\vec{J} = \int d^3x\, \psi^\dagger \{\vec{x} \times (-i\vec{\nabla}) + \tfrac{1}{2}\vec{\Sigma}\}\psi}$$

now consider particles at rest: $a^{s\dagger}_{\vec{p}=0}|0\rangle$ [and $b^{s\dagger}_{\vec{p}=0}|0\rangle$]

$J_z|0\rangle = 0$

$$\rightarrow J_z a^{s\dagger}_0 |0\rangle = [J_z, a^{s\dagger}_0]|0\rangle$$

$$\vdots$$

$$= \frac{1}{2m}\sum_r u^{r\dagger}(\vec{0})\frac{\Sigma^3}{2}u^s(\vec{0})a^{r\dagger}_0|0\rangle$$

$u(0) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$

$v(0) = \sqrt{m}\begin{pmatrix}\eta\\ -\eta\end{pmatrix}$

$$= \sum_r \xi^{r\dagger} \frac{\sigma^3}{2} \xi^s a_0^{r\dagger} |0\rangle$$ | choose $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

(eigenstates of σ^3)

$$\Rightarrow \boxed{\boxed{\begin{aligned} J_z a_0^{s\dagger} |0\rangle &= \pm \frac{1}{2} a_0^{s\dagger} |0\rangle \\ J_z b_0^{s\dagger} |0\rangle &= \mp \frac{1}{2} b_0^{s\dagger} |0\rangle \end{aligned}}}$$

upper sign $\xi/\eta = \begin{pmatrix}1\\0\end{pmatrix}$

lower sign $\xi/\eta = \begin{pmatrix}0\\1\end{pmatrix}$

⌜similar (formal) argument for Klein Gordon particles: "$\mathbb{1}_{1/2} = \mathbb{1}$" (scalar) $\rightarrow$ no term like $\vec{\Sigma}$

$\rightarrow \ldots \rightarrow J_z a_0^\dagger |0\rangle = 0 \Leftrightarrow$ spin $= 0$ ⌟

Discrete symmetries C, P, T

- parity P

classical : $\psi \xrightarrow{\vec{x} \to -\vec{x}} \Lambda_{1/2}\, \psi$ $\quad \Lambda_{1/2}(P) = \eta\, \gamma^0$

QM : $\psi |0\rangle \longrightarrow P\psi|0\rangle = \underbrace{P\psi P^{-1}}_{\psi' = P\psi P} P|0\rangle$

want : $P a^s_{\vec{p}} P = \eta_a\, a^s_{-\vec{p}}$ $\qquad P b^s_{\vec{p}} P = \eta_b\, b^s_{-\vec{p}}$

phases : observables always contain an even number of operators, and should remain unchanged after applying P twice

$\Rightarrow \eta_a^2 = \pm 1 \;;\; \eta_b^2 = \pm 1$

(see P&S)

$$\boxed{P\psi(t,\vec{x})P = \eta_a \gamma^0 \psi(t,-\vec{x})} \quad \text{and} \quad \boxed{\eta_a \cdot \eta_b = -1}$$

$\bar{\psi}$ $\qquad \eta_a^* \bar{\psi}(t,\vec{x})\gamma^0$

$\rightsquigarrow$ can set $\eta_a = -\eta_b [= 1]$

$\Rightarrow P a^{s+}_{\vec{p}} b^{r+}_{\vec{q}} |0\rangle = - a^{s+}_{-\vec{p}} b^{r+}_{-\vec{q}} |0\rangle$

time reversal T

flipped spin

want: $a_{\vec{p}} \longrightarrow a_{-\vec{p}}$

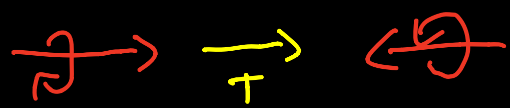
$\wedge$ $\psi(t,\vec{x}) \to \psi(-t,\vec{x})$

$\wedge$ spin flip!

$$\Rightarrow T a^{s}_{\vec{p}} T = a^{-s}_{-\vec{p}}$$

$$\boxed{T\, i\, T = -i}$$

T is "anti-unitary"



(P&S)

$$\boxed{T\psi(t,\vec{x})T = \gamma^1\gamma^3\psi(-t,\vec{x})}$$

Charge conjugation C

(fermion $\leftrightarrow$ antifermion w/ same spin, momentum)

$$C a^{s}_{\vec{p}} C = b^{s}_{\vec{p}}$$

$$C b^{s}_{\vec{p}} C = a^{s}_{\vec{p}}$$

$$\rightsquigarrow \quad \boxed{C\psi(x)C = -i\gamma^2\psi^*(x)}$$

C P T

explicit representations of C, P, T allow to work out transformation of $\bar\psi \Gamma \psi$:

	$\bar\psi\psi$	$i\bar\psi\gamma^5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	$\bar\psi\sigma^{\mu\nu}\psi$	$i\partial_\mu$
P	+	−	$(-1)^\mu = \begin{cases} + & \text{for } \mu=0 \\ - & \mu=i \end{cases}$	$-(-)^\mu$	$(-)^\mu(-)^\nu$	$(-)^\mu$
T	+	−	$(-)^\mu$	$(-)^\mu$	$-(-)^\mu(-)^\nu$	$+(-)^\mu$
C	+	+	−	+	−	$+ \to -$
CPT	+	+	−	−	+	−

Dirac Lagrangian: $\mathcal{L}_{Dirac} = \bar\psi(i\not\partial - m)\psi$

C, P, T ✓

<u>CPT theorem</u> (Pauli) : Any QFT that satisfies the following is invariant under CPT:

- Lorentz invariance
- causality
- locality
- Hamiltonian bounded from below ("$E_p > 0$")

even stronger: ~~CPT~~ => ~~Lorentz invariance~~

6. Perturbation theory

$$\mathcal{L}_0 \longrightarrow \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

free fields: quadratic $\Rightarrow$ linear equations of motion

non-linear terms! $\Rightarrow$ couple different Fourier modes

$$\Rightarrow H_{int}[\phi] = \int d^3x \, \mathcal{H}_{int}[\phi, \partial\phi] = -\int d^3x \, \mathcal{L}_{int}[\phi, \partial\phi]$$

NB: • these are <u>local</u> interactions

[e.g. $\phi^2(x)\phi(y)$ not allowed]

• $\partial\phi$ dependence changes def. of π!

$$\Rightarrow \boxed{\partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}_0}{\partial \phi} = \underbrace{\frac{\partial \mathcal{L}_{int}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_{int}}{\partial(\partial_\mu \phi)}}_{\text{inhomogeneous term}}} \quad (\ast)$$

$\rightsquigarrow$ Green's functions!

simplest example : "ϕ^4-theory" (~ Higgs !)

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!} \phi^4}_{\mathcal{L}_{int}}$$

$$(*) \Rightarrow \quad (\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

perturbation theory : $\lambda \ll 1$

$$\Rightarrow \phi = \phi_0 + \lambda \phi_1 + \lambda^2 \phi_2 + \dots$$

where $(\partial^2 + m^2)\phi_0 = 0$

$$(\partial^2 + m^2)(\cancel{\phi_0} + \cancel{\lambda}\phi_1 + \dots) = -\frac{\cancel{\lambda}}{3!}(\phi_0 + \dots)$$

$\vdots$

correlation functions

⇝ fundamental "building blocks" to describe (not only) interactions!

simplest example: 2-point function = Green's function:

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \underbrace{\langle 0 | T \phi(x) \phi(y) | 0 \rangle_{free}}_{= D_F(x-y)} + \mathcal{O}(\lambda)$$

ground state of interacting theory

goal: compute expansion in λ!

step 1: express $\phi(x)$ in terms of free-field solutions

1a) transform to interaction picture:

$$\phi(t, \vec{x}) = e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH(t-t_0)} e^{-iH_0(t-t_0)}}_{= U^+(t,t_0)} \underbrace{e^{+iH_0(t-t_0)} \phi(t_0, \vec{x}) e^{-iH_0(t-t_0)}}_{\equiv \phi_I(t, \vec{x})} \underbrace{e^{+iH_0(t-t_0)} e^{-iH(t-t_0)}}_{\equiv U(t,t_0)}$$

$\boxed{\text{"interaction picture field"}}$

$(\neq 1)$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^{*} e^{ipx} \right) \Big|_{x^0 = t - t_0}$$

⌈possible because H_0 can be diagonalized as before⌋

(*) $U = (t, t_0)$: "time evolution operator" / "interaction picture propagator"

1b) determine U in terms of ϕ_I

$$\partial_t U(t,t_0) = i H_0 e^{iH_0(t-t_0)} e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}(-iH) e^{-iH(t-t_0)}$$

$$= -i\, e^{iH_0(t-t_0)} \underbrace{(H - H_0)}_{= H_{int}} e^{-iH(t-t_0)}$$

$$= \underbrace{-i\, e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}_{\equiv H_I = H_{int}[\phi_I] \left(= \frac{\lambda}{4!}\int d^3x\, \phi_I^4\right)} \cdot U(t,t_0)$$

⌈because $e^{iH_0} \phi^n e^{-iH_0} = \underbrace{e^{iH_0}\phi e^{-iH_0}}_{\phi_I} e^{iH_0} \dots \underbrace{e^{iH_0}\phi e^{-iH_0}}_{\phi_I}$
$= \phi_I^n$ ⌋

$\Rightarrow$ solution : $U(t,t_0)$ "=" $\cancel{e^{-i\int_{t_0}^{t} H_I\, dt}}$

not true because $[H_I, U] \neq 0$!

$$= 1 + (-i)\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \underbrace{H_I(t_1) H(t_2)}_{\text{do not}}$$

(in general)
commute!

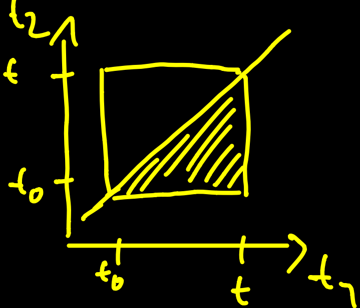
$$+ (-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \, H_I(t_1) H_I(t_2) H_I(t_3) + \dots$$

apply ∂_t : each term $= -i H_I \times$ previous term ✓

now simplify, noting that all terms are time-ordered:

$$\Rightarrow \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n \, H_I(t_1) \dots H_I(t_n)$$

$$= \underbrace{\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2}_{} \dots \int_{t_0}^{t_{n-1}} dt_n \, T\{H_I(t_1) \dots H_I(t_n)\}$$



$$\Rightarrow \underbrace{\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2}_{} \, f(t_1, t_2) = \frac{1}{2} \underbrace{\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2}_{} \, f(t_1, t_2)$$

if $f(t_1, t_2) = f(t_2, t_1)$

$$= \frac{1}{n!} \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n \, T\{H_I(t_1) \dots H_I(t_n)\}$$

$$\Rightarrow \boxed{U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt'\, H_I(t')\right]\right\}} \quad \text{for any } t \geq t_0$$

$\Rightarrow$ (...)

- $U^\dagger = U^{-1}$
- $U^{-1}(t_1,t_2) = U(t_2,t_1)$
- $U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$ for $t_1 \geq t_2 \geq t_3$

<u>step 2</u>: express $|\Omega\rangle$ in terms of free-field quantities

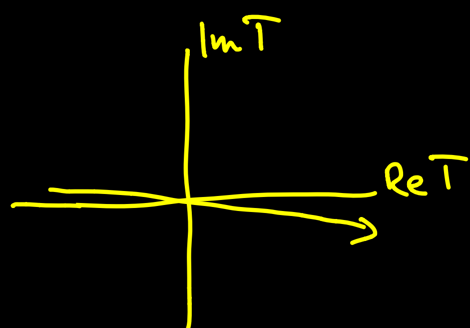
consider $e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$

eigenvalues and -states of H

$$\overset{(*)}{=} e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

- $\langle\Omega|H|\Omega\rangle < E_n \;\; \forall\, n \neq 0$!
- $\langle\Omega|0\rangle \neq 0$ (by <u>assumption</u> of small perturbation!)

now take limit $T \to \infty(1-i\varepsilon)$



(#)

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT} \underbrace{|0\rangle}_{=|0\rangle} \quad \Big| T\to T+t_0$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE(T+t_0)}\langle\Omega|0\rangle\right)^{-1} \underbrace{e^{iH(-T-t_0)} \overbrace{e^{-iH_0(-T-t_0)}}^{}}_{=U^{-1}(-T,t_0)=U(t_0,-T)} |0\rangle$$

similar: $\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$

$$\Rightarrow \langle\Omega|T\{\phi(x)\dots\phi(y)\}|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1}$$

$$\times \underbrace{\langle 0|U(T,t_0)\,T\{\phi(x)\dots\phi(y)\}\,U(t_0,-T)|0\rangle}_{} \quad \Big| \phi(x) = U(t_0,x^0)\,\phi_I(x)\,U(x^0,t_0)$$

$$\langle 0|T\{\underbrace{U(T,t_0)U(t_0,x^0)}_{=U(T,x^0)}\phi_I(x)\underbrace{U(x^0,t^0)\dots U(t_0,y^0)}_{U(x^0,y^0)}\phi_I(y)\underbrace{U(y^0,t_0)U(t_0,-T)}_{U(y^0,-T)}\}|0\rangle$$

$$= \langle 0|T\{\phi_I(x)\dots\phi_I(y)\,U(T,-T)\}|0\rangle$$

$$\Rightarrow \boxed{\langle\Omega|T\{\phi(x)\ldots\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\ldots\phi_I(y)\exp[-i\int_{-T}^{T}dt\,H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T}dt\,H_I(t)]\}|0\rangle}}$$

NB : exact expression, but suitable for expansion in small couplings ($H_I \propto \lambda$)

$H_I \sim \phi^n$

$\Rightarrow$ need only ever to evaluate

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\ldots\phi(x_m)\}|0\rangle \;!$$

Wick's theorem

goal : simplify calculations of $\langle 0 | T\{ \dots \} | 0 \rangle$

NB : drop index "I" in the following, i.e. $\phi_I(x) \to \phi(x)$
(we are always in the interaction picture!)

$$\phi(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx}}_{\equiv \phi^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{+ipx}}_{\phi^-(x)}$$

Def. <u>normal order</u> $N(\mathcal{O})$ of an operator $\mathcal{O}$:
place all $a^\dagger / \phi^-$ to the left
a / ϕ^+ to the right

$$\Rightarrow \langle 0 | N(\mathcal{O}) | 0 \rangle = 0$$

sometimes ":$\mathcal{O}$:" is also used

Def. <u>contraction</u>

$$\phi(x)\phi(y) = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases}$$

$$= D_F(x-y)$$

Wick's theorem

$$\boxed{\begin{aligned} &T\{\phi(x_1)\dots\phi(x_n)\} \\ &= N\{\phi(x_1)\dots\phi(x_n) + \text{all possible contractions}\} \end{aligned}}$$

$$\Rightarrow \boxed{\begin{aligned} &\langle 0|T\{\phi(x_1)\dots\phi(x_n)\}|0\rangle \\ &= \sum \text{all } \underline{\text{full}} \text{ contractions} \end{aligned}}$$

⌈proof by induction: a) for $n=2$

b) show $n-1 \Rightarrow n$ (P&S)

a) $$T\{\underbrace{\phi(x_1)}_{\equiv\phi_1}\underbrace{\phi(x_2)}_{\equiv\phi_2}\} = T\{\phi_1^+\phi_2^+\} + T\{\phi_1^-\phi_2^-\} + \underbrace{T\{\phi_1^+\phi_2^-\}}_{\substack{=N\{\phi_1^+\phi_2^-\} \\ \text{if } x_1^0 < x_2^0}} + \underbrace{T\{\phi_1^-\phi_2^+\}}_{\substack{=N\{\phi_2^+\phi_1^-\} \\ \text{if } x_2^0 < x_1^0}}$$

$$= \underbrace{\phi_1^+\phi_2^+ + \phi_1^-\phi_2^- + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^+}_{=N\{\phi_1\phi_2\}} + \underbrace{\begin{cases} [\phi_1^+, \phi_2^-] & \text{for } x_1^0 > x_2^0 \\ [\phi_2^+, \phi_1^-] & \text{for } x_2^0 > x_1^0 \end{cases}}_{\overline{\phi_1\phi_2}}$$

⌋

example 1

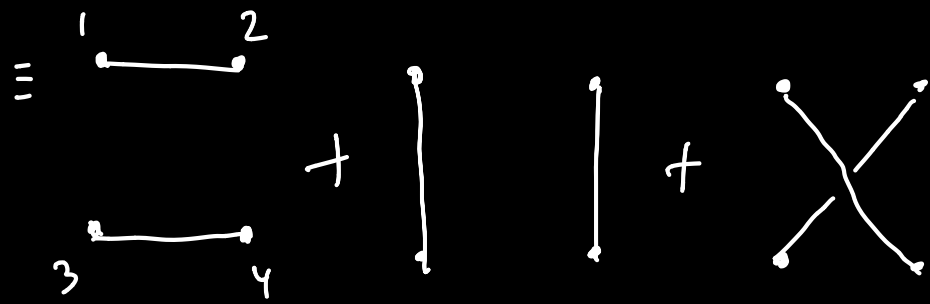
$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \wick{\c\phi_1\c\phi_2}\phi_3\phi_4 + \wick{\c\phi_1\phi_2\c\phi_3}\phi_4 + \wick{\c\phi_1\phi_2\phi_3\c\phi_4}$$

$$+ \phi_1\wick{\c\phi_2\c\phi_3}\phi_4 + \phi_1\wick{\c\phi_2\phi_3\c\phi_4} + \phi_1\phi_2\wick{\c\phi_3\c\phi_4}$$

$$+ \wick{\c\phi_1\c\phi_2\c\phi_3\c\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4}\}$$

$$\Rightarrow \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = D_F(x_1-x_2)\,D_F(x_3-x_4)$$
$$+ D_F(x_1-x_3)\,D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)\,D_F(x_2-x_3)$$

$\equiv$



"Feynman diagrams"

example 2 :

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle \propto \langle 0|T\{\phi(x)\phi(y)\exp[-i\int_{-T}^{T} d^4z\,\frac{\lambda}{4!}\phi^4(z)]\}|0\rangle$$

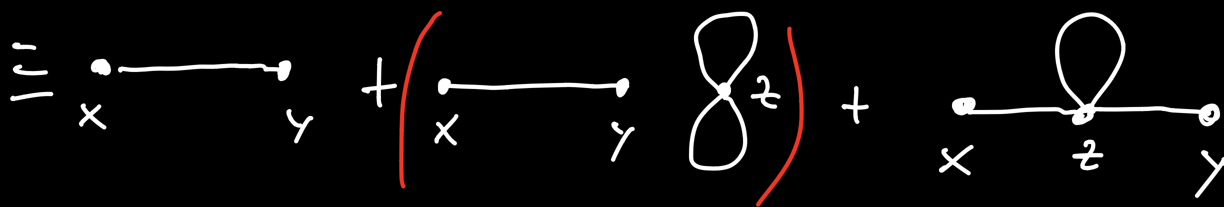
$$= \underbrace{\langle 0|T\{\phi(x)\phi(y)\}|0\rangle}_{D_F(x-y)} - i\frac{\lambda}{4!}\int d^4z\,\langle 0|T\{\phi(x)\phi(y)\phi(z)^4\}|0\rangle$$

$$\phi(z)^4 = \phi_z\phi_z\phi_z\phi_z$$

$$= D_F(x-y) - i\frac{\lambda}{4!}\int d^4z\, D_F(x-y)\,D_F(z-z)^2 \times 3 \quad (3 \text{ possibilities to}$$

$$-i\frac{\lambda}{4!}\int d^4z\, D_F(x-z)\,D_F(y-z)\,D_F(z-z)\times 4\times 3$$

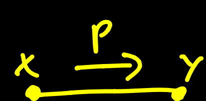
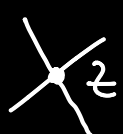
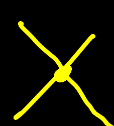
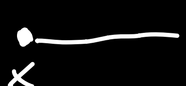
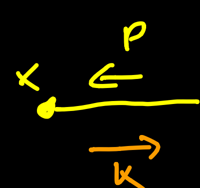
contract ϕ_z^4)



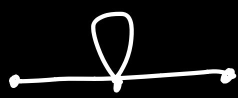
Feynman rules for ϕ^4 theory

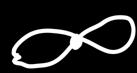
$$\boxed{\langle 0|T\{\phi(x_1)\dots\phi(x_n)\exp[-i\int dt\, H_I(t)]\}|0\rangle = \text{Sum of all possible diagrams with } n \text{ external points}}$$

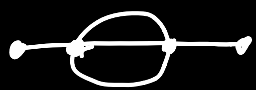
where (for ϕ^4 theory)

position space	momentum space
1. for each propagator  $= D_F(x-y)$	 $= \frac{i}{p^2-m^2+i\epsilon}$
2. for each "vertex" (internal points)  $= (-i\lambda)\int d^4z$	 $= -i\lambda$
3. for external point:  $= 1$	 $= e^{-ipx}$, e^{+ipx}

4. Divide by symmetry factor
$\equiv$ number of ways of interchanging components without changing the diagrams

e.g.  $S=2$ $(z \leftrightarrow z)$

 $S = 2^3 = 8$

 $S = 3! = 6$

⋮

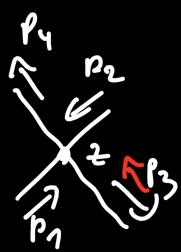
in case of doubt: count equivalent contractions!

5. Impose 4-momentum conservation @ each vertex

6. integrate over all (undetermined) momenta

Feynman rules in momentum space

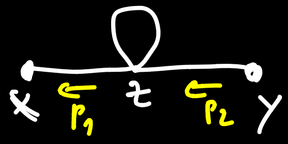
$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)} = D_F(x-y)$$

vertex: 

$$= -i\lambda \int d^4z \; e^{-ip_1 z} e^{-ip_2 z} e^{\mp i p_3 z} e^{+ip_3 z}$$

$$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 \pm p_3 - p_4)$$

example :



position space : $\frac{1}{2}(-i\lambda)\int d^4z\, D_F(x-z)\, D_F(z-y)\, D_F(z-z)$

$$= \frac{1}{2}(-i\lambda)\int d^4z \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4p_3}{(2\pi)^4} \times$$

$$\times \frac{i}{p_1^2-m^2+i\epsilon}\,\frac{i}{p_2^2-m^2+i\epsilon}\,\frac{i}{p_3^2-m^2+i\epsilon} \times$$

$$\times\, e^{-ip_1(x-z)}\, e^{-ip_2(z-y)}\, \cancel{e^{-ip_3(z-z)}}$$

$$= \underbrace{\frac{1}{2}}_{4.}\underbrace{(-i\lambda)}_{2.}\int \underbrace{\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_3}{(2\pi)^4}}_{6.}\;\underbrace{\frac{i}{p_1^2-m^2+i\varepsilon}\cdots\frac{i}{p_3^2-m^2+i\varepsilon}}_{1.} \times$$

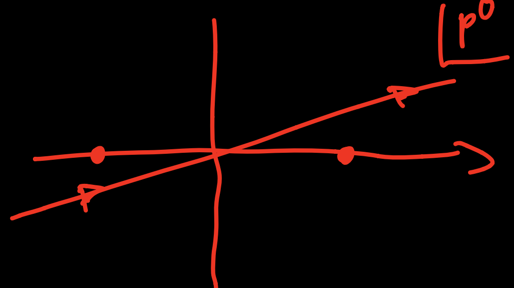
$$\times\, \underbrace{e^{-ip_1x}\, e^{+ip_2y}}_{3.}\;\underbrace{(2\pi)^4\,\delta^{(4)}(p_1-p_2)}_{5.}$$

NB : • $\int d^4z = \lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \int d^3z$

• $e^{ip\cdot z} \Rightarrow p\cdot z \supset p^0 z^0$ must be real

$\Rightarrow p^0 \propto (1+i\varepsilon)$

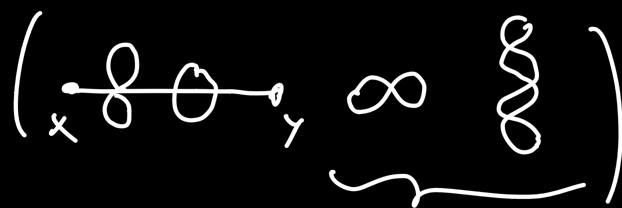
$\Rightarrow$



i.e same pole prescription as for Feynman propagator! ✓

Exponentiation of disconnected diagrams

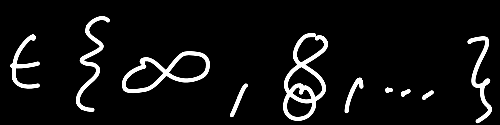
typical diagram:

$\langle 0 | T\{\phi(x)\phi(y)\exp[-i\int dt\, H_I(t)]\} | 0\rangle \supset$ 

"disconnected pieces"

= no connection to external points (x or y)

label all disconnected pieces:

$V_i \in$ 

$\Rightarrow$ every diagram = (value of connected piece)

$$\times \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

symmetry factor = number of possibilies of arranging n_i identical pieces

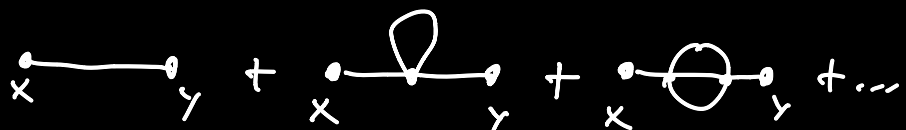
$$\Rightarrow \sum \text{all diagrams} = \sum_{\substack{\text{all possible}\\ \text{connected}\\ \text{pieces}}} \sum_{\{n_1, n_2, n_3, \dots\}} \begin{pmatrix}\text{value of}\\ \text{conn. piece}\end{pmatrix} \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

$$= \left(\sum \text{connected}\right) \times \underbrace{\sum_{\{n_1, n_2, \dots\}} \prod_i \frac{1}{n_i!} (V_i)^{n_i}}_{\prod_i \sum_{n_i=1}^{\infty} \frac{1}{n_i!}(V_i)^{n_i}}$$

$$= \prod_i \exp V_i = \exp \sum_i V^i$$

e.g. 2-point function:

$$\langle 0 | T\{\phi(x)\phi(y) \exp[-i\int dt\, H_I(t)]\}|0\rangle$$

= ()

× exp [] } "energy density of vacuum"

$$\Downarrow$$

$$\boxed{\begin{aligned}\langle \Omega | T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x_1)\dots\phi_I(x_n)\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle} \\ &= \begin{pmatrix}\text{sum of all } \underline{\text{connected}} \text{ diagrams}\\ \text{with } n \text{ external points}\end{pmatrix}\end{aligned}}$$

7. Cross sections and decay rates

cross section σ ~ effective target area seen by an interacting particle

$\propto$ probability for interaction to happen

introduce in 3 steps...

i)

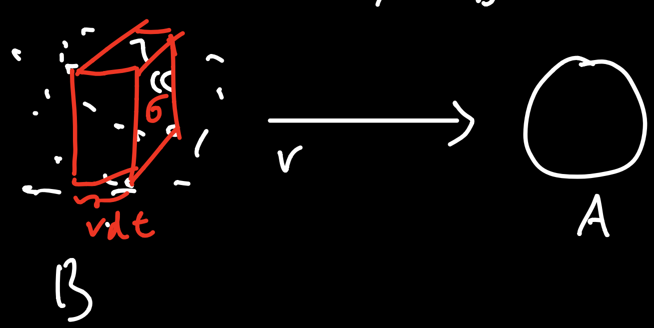
"hard sphere approximation":

scattering takes place if $b < r$

no scattering if $b > r$

$\Rightarrow \sigma = \pi b^2$

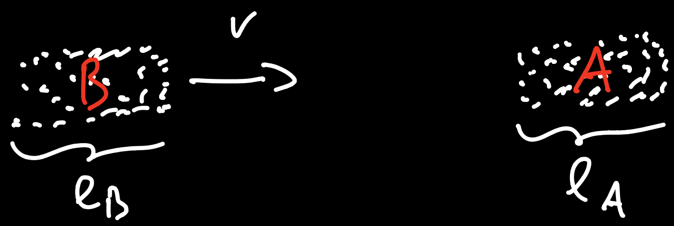
ii) many incident particles (number density n_B)



$\Rightarrow$ # events $= n_B \cdot \sigma \, v dt$

$$\rightsquigarrow \boxed{\sigma \equiv \frac{\#\text{events}/\text{time}}{v \cdot n_B}}$$

iii) two colliding beams:



$\Rightarrow$ # events $\overset{!}{\propto}$ $l_B\, l_A\, n_B\, n_A\, A$ ← overlapping area of two beams

$$\rightsquigarrow \boxed{\sigma \equiv \frac{\#\text{ events}}{n_A n_B l_A l_B A} = A\,\frac{\#\text{ events}}{N_A N_B}}$$

- $N_A = 1$
- $N_B = n_B \cdot A \cdot \underbrace{l_B}_{v \cdot dt}$

$\rightsquigarrow$ case ii) ✓

decay rate

$$\Gamma \equiv \frac{\#(\text{decays/time})}{\#\text{ particles still left}}$$

$$= -\frac{dn/dt}{n} \Rightarrow n = n_0\, e^{-\Gamma t}$$

$\rightsquigarrow$ lifetime $\tau \equiv \frac{1}{\Gamma}$

The S-matrix

general wave packet : $|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(\vec{k}) \, |\vec{k}\rangle$

one-particle states of momentum $\vec{k}$ in interacting (!) theory

⇝ want to compute "scattering probability" for $A + B \longrightarrow \ldots$:

$$P = |\underbrace{\langle \phi_1 \phi_2 \ldots|}_{\text{"out-state"}} \underbrace{|\phi_A \phi_B\rangle}_{\text{"in-state"}}|^2$$

set up in distant future

set up in remote past

BUT: still in Heisenberg picture! [states are t-independent, but operators – and hence their eigenvalues, like $\vec{k}$ – are time-dependent!]

in-state : will consider highly concentrated wave packages "particles"

out-state : – –   plane waves

~ what is measured by detectors

Definition of "$\underline{S\text{-matrix}}$"

$$\boxed{{}_{\text{out}}\langle \vec{p}_1 \vec{p}_2 \cdots | \vec{k}_A \vec{k}_B \rangle_{\text{in}} \equiv \langle \vec{p}_1 \vec{p}_2 \cdots | S | \vec{k}_A \vec{k}_B \rangle}$$

constructed @ any common reference time

expected structure:

$$S = 1 + iT$$

no scattering

"T-matrix" $\propto \delta^{(4)}(k_A + k_B - \sum_f p_f)$

$\Rightarrow$ Def. "invariant matrix element $\mathcal{M}$": [~ scattering amplitude in QM]

$$\boxed{\langle \vec{p}_1 \vec{p}_2 \cdots | iT | \vec{k}_A \vec{k}_B \rangle \equiv (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_f p_f) \cdot i\mathcal{M}(k_A k_B \to p_f)}$$

from $\mathcal{M}$ to σ, Γ

consider [single] target A, many incident particles B:

$\Rightarrow$ initial state:

$$|\phi_A \phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(\vec{k}_A)\,\phi_B(\vec{k}_B)}{2\sqrt{E_A E_B}} e^{i\vec{k}_B\cdot\vec{b}} |k_A k_B\rangle$$

number of scattering events:

$$N = \underbrace{\int d^2b \frac{N_B}{A_{rea}}}_{\substack{\text{\# particles} \\ \text{within range} \\ \text{of } b}} P(AB \to 1,2,\dots,n)$$

assume constant over range of interaction

$$= \frac{N_B}{A_{rea}} \int d^2b \underbrace{\prod_{f=1}^{n} \int \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}}_{\substack{\text{"sum over all possible} \\ \text{momentum configurations} \\ \text{in final state"}}} \left|{}_{out}\langle p_1 p_2 \cdots p_n | \phi_A \phi_B\rangle\right|^2$$

$$\Rightarrow d\sigma = \frac{A_{rea}}{N_B} \frac{dN}{(N_A=1)} = \left(\prod_f \frac{d^3p}{(2\pi)^3} \frac{1}{2E_f}\right) \int d^2b \prod_{i=A,B} \int \frac{d^3k_i}{(2\pi)^3} \frac{\phi_i(\vec{k}_i)}{\sqrt{2E_i}}$$

w.r.t. final configuration

$$\int \frac{d^3k_i'}{(2\pi)^3} \frac{\phi_i(\vec{k}_i')}{\sqrt{2E_i'}} \underbrace{e^{i\vec{b}\cdot(\vec{k}_B' - \vec{k}_B)}}_{\int d^2b \to (2\pi)^2 \delta^{(2)}(\vec{k}_B^{\perp} - \vec{k}_B'^{\perp})} \times$$

$$\times {}_{\text{out}}\langle \vec{p}_1 \cdots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{\text{in}} \left({}_{\text{out}}\langle \vec{p}_1 \cdots p_n | \vec{k}_A' \vec{k}_B' \rangle_{\text{in}} \right)^*$$

$$i\mathcal{M}(\vec{k}_A \vec{k}_B \to \{\vec{p}_f\})(2\pi)^4 \delta^{(4)}(k_A + k_B - \sum_f p_f)$$

- 12 integrals, 10 δ-functions
- recall that ϕ_i are highly localized in $\vec{k}$-space $\rightsquigarrow$ pull outside integrals
- ...

$$\boxed{d\sigma = \frac{1}{2E_A\, 2E_B |v_A - v_B|}\, d\Pi_n\, |\mathcal{M}(p_A p_B \to \{p_f\})|^2}$$

"relativistically invariant n-body phase space"

$$\boxed{d\Pi_n = \left(\prod_f \frac{d^3p}{(2\pi)^3} \frac{1}{2E_f}\right)(2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f)}$$

↳ purely kinematical object, can be computed "once and for all"

[$\mathcal{M}$ encodes the dynamics, i.e. the model-dependent part of the interactions]

prefactor: $(E_A E_B |v_A - v_B|)^{-1} = |E_B k_A - E_A k_B|^{-1} = |\varepsilon_{\mu x y \nu} K_A^\mu K_B^\nu|^{-1}$

(...) ⇝ same transformation properties as an area in z-direction!

(e.g. invariant und boost along z-direction)

similar: decay rate $\boxed{d\Gamma = \frac{1}{2m_A}\, d\Pi_n\, |\mathcal{M}(m_A \to \{p_f\})|^2}$

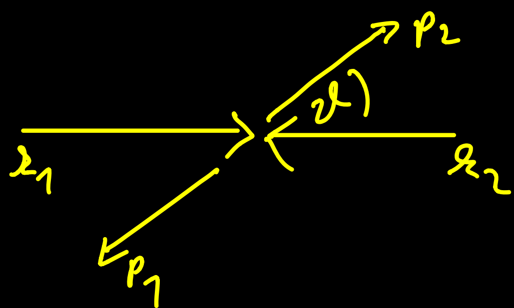
How to treat identical final-state particles?

a) restrict $\int d\Pi_n$ to physically inequivalent configurations

b) integrate over all sets of $\{\vec{p}_f\}$ and then divide by $(n!)$.

example: 2-body final state in center-of-mass system [CMS]

- $\sum \vec{k}_i + \sum \vec{p}_f = 0$
- $\sum k_i^0 = \sum p_f^0 \equiv E_{cm}$



$$\Rightarrow \int d\Pi_2 = \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{2E_2}(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2)$$

$$= \int \frac{\overbrace{d^3p_1}^{=p_1^2\, d\Omega\, dp_1}}{(2\pi)^3}\frac{1}{2E_1}\frac{1}{\underbrace{2E_2}_{=E_2(\vec{k}_1,\vec{k}_2,\vec{p}_1)}}(2\pi)\,\delta(E_{cm}-E_1-E_2)$$

- $E_i = \sqrt{|\vec{p}_i|^2 + m_i^2}$
- $\delta(f(x)) = \frac{1}{|f'(x_0)|}\,\delta(x-x_0)$

$$= \int \frac{d\Omega}{(2\pi)^2} \frac{|\vec{p}_1|^2}{4E_1E_2} \left| \underbrace{-\frac{dE_1}{dp_1}}_{= -\frac{|\vec{p}_1|}{E_1}} - \underbrace{\frac{dE_2}{dp_1}}_{-\frac{|\vec{p}_1|}{E_2}} \right|^{-1}$$

$$= \int \frac{d\Omega}{(2\pi)^2} \frac{|\vec{p}_1|}{4} \left| \underbrace{E_1 + E_2}_{E_{cm}} \right|^{-1}$$

$$\Rightarrow \boxed{\int d\Pi_2 = \int d\Omega \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{E_{cm}}} \quad \text{in CMS frame}$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{4E_AE_B|v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4E_{cm}} |M|^2$$

if $|M|^2$ is symmetric about collision axis :

$$\int d\Pi_2 = \int d\cos\vartheta \frac{1}{8\pi} \frac{\vec{p}_1}{E_{cm}}$$

$$= \begin{cases} \int_{-1}^{+1} & \text{for distinguishable final-state particles} \\ \int_{0}^{+1} & \text{" identical " "} \end{cases}$$

Calculating $\mathcal{M}$ from Feynman diagrams

claim: S-matrix is "simply the Fourier transform of an n-point correlation function"

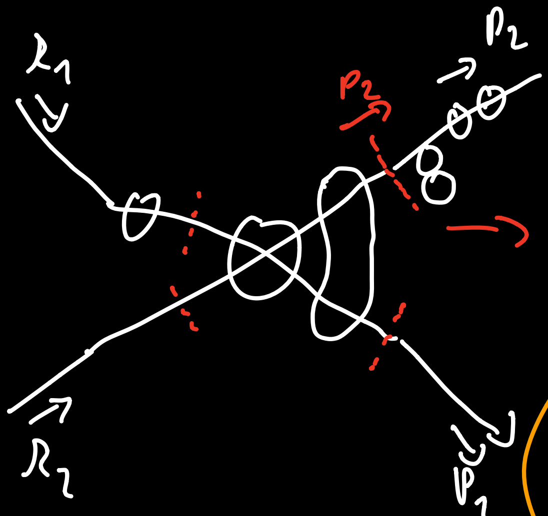
⤳ "LSZ reduction formula"

[Lehman, Symanzik & Zimmermann
proof → QFT 2 !]

$$\boxed{\begin{gathered}\prod_{i=1}^{n}\int d^4x_i\, e^{ip_i x_i}\prod_{j=1}^{m}\int d^4y_j\, e^{-ik_j y_j}\langle\Omega|T\{\overbrace{\phi(x_1)\ldots\phi(x_n)}^{\text{"out"}}\overbrace{\phi_1(y_1)\ldots\phi(y_m)}^{\text{"in"}}\}|\Omega\rangle \\ \xrightarrow[\substack{p_i^0\to E_{\vec p_i}\\ k_i^0\to E_{\vec k_i}}]{}\left(\prod_{i=1}^{n}\frac{\sqrt{Z}\,i}{p_i^2-m^2+i\epsilon}\right)\left(\prod_{j=1}^{m}\frac{\sqrt{Z}\,i}{k_j^2-m^2+i\epsilon}\right)\langle\vec p_1\ldots\vec p_n|S|\vec k_1\ldots k_m\rangle\end{gathered}}$$

(@ any common reference time)

consider an individual diagram [in ϕ^4 theory]



$\longrightarrow \sim \left.\frac{i}{p_2^2-m^2}\right|_{p_2^2=m^2}$

~ describes $|\vec p\rangle_0 \longrightarrow |\vec p\rangle$

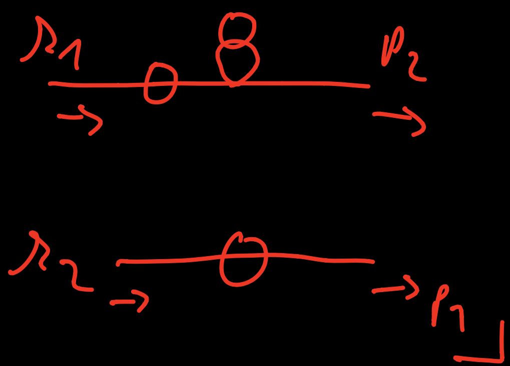
⤳ nothing to do with

actual scattering process

$$\Rightarrow \boxed{i\mathcal{M} = \text{sum of all } \underline{\underline{\text{fully}}} \text{ connected, } \underline{\text{amputated}} \text{ diagrams}}$$

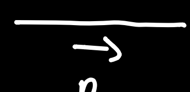
$S = 1 + i\underline{\underline{T}}$

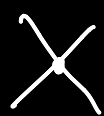
⌈i.e. do not include, e.g.,



⌋

$\rightsquigarrow$ rules: ⌈for $i\mathcal{M}$!⌋

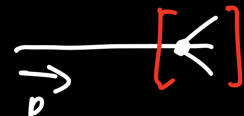
1. propagator  $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

2. vertex  $= -i\lambda$

   & impose 4-momentum conservation @ each vertex

   & integrate over each undetermined (= loop!) momenta

3. divide by symmetry factor

! 4. external lines  $= 1$

[$\Leftrightarrow$ points for correlation functions!]

Motivation for LSZ:

$$\langle \vec{p}_1 \vec{p}_2 \dots | S | \vec{k}_A \vec{k}_B \rangle = {}_{\text{out}}\langle \vec{p}_1 \vec{p}_2 \dots | k_A k_B \rangle_{\text{in}}$$

$$= \lim_{T \to \infty} \langle \vec{p}_1 \vec{p}_2 \dots | e^{-iH(2T)} | \vec{k}_A \vec{k}_B \rangle$$

recall: $|\Omega\rangle \propto \lim_{T \to \infty(1-i\varepsilon)} e^{-iHT} |0\rangle$

$$\text{"}\Rightarrow\text{"} \quad |k_A k_B\rangle \propto \lim_{T \to \infty(1-i\varepsilon)} e^{-iHT} |k_A k_B\rangle_0$$

$$\Rightarrow \langle \vec{p}_1 \vec{p}_2 \dots | S | \vec{k}_A \vec{k}_B \rangle \propto \lim_{T \to \infty(1-i\varepsilon)} {}_0\langle \vec{p}_1 \vec{p}_2 | T\left\{ \exp\left[-i \int_{-T}^{T} dt\, H_I(t) \right] \right\} | \vec{k}_A \vec{k}_B \rangle_0$$

Summary "QFT in a nutshell"

$$\langle \vec{p}_1 \vec{p}_2 \dots | iT | \vec{k}_A \vec{k}_B \rangle \equiv i\mathcal{M} (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_f)$$
$$= {}_0\langle \vec{p}_1 \vec{p}_2 \dots | T\{\exp[-i \int_{-T}^{T} dt\, H_I(t)]\} | \vec{k}_A \vec{k}_B \rangle_0 \Big|_{\#}$$

fully connected + amputated

1. Expand exp[..] in coupling constant(s)
2. Use Wick's theorem to expand T{...}
3. contract every external state w/ one operator from the expansion
4. Contract all remaining operator w/ each other
5. Disregard amplitudes that can be "amputated"

[= any of the propagators is on shell]

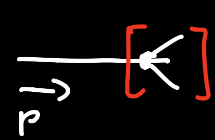
<u>example</u>: $\mathcal{O}(\lambda)$ contribution to $\langle \vec{p}_1 \vec{p}_2 | iT | \vec{k}_A \vec{k}_B \rangle$

$$\to {}_0\langle \vec{p}_1 \vec{p}_2 | -i \frac{\lambda}{4!} T\{ \int d^4x\, \phi_I(x)^4 \} | \vec{k}_A \vec{k}_B \rangle$$

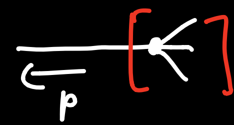
$$= -i \frac{\lambda}{4!} \int d^4x\, {}_0\langle \vec{p}_1 \vec{p}_2 | N\{\phi(x)\phi(x)\phi(x)\phi(x) + \text{all possible contractions}\} | \vec{k}_A \vec{k}_B \rangle$$

e.g. $\phi^+(x) |\vec{p}\rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} a_k e^{ikx} \sqrt{2E_p}\, a_p^+ |0\rangle$
$= e^{-ipx} |0\rangle$

$\Rightarrow$ $\phi(x)\,|\vec{p}\rangle_0 \equiv e^{-ipx}\,|0\rangle$

$\hat{=}$  $= 1$

${}_0\langle\vec{p}\,|\,\phi(x) \equiv {}_0\langle 0|\,e^{+ipx}$

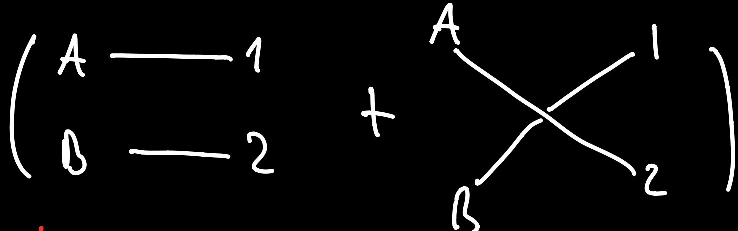
$\hat{=}$  $= 1$

NB: In total, only equal numbers of $a^\dagger$ and a survive in $\langle \vec{p}_1 \vec{p}_2 \ldots |\, \phi^m \,|\, \vec{k}_A \vec{k}_B\rangle \sim \langle 0|(a)^n (\phi^+ + \phi^-)^m (a^\dagger)^2 |0\rangle$

$\rightsquigarrow$ every ϕ must be "contracted" with either initial or final state!

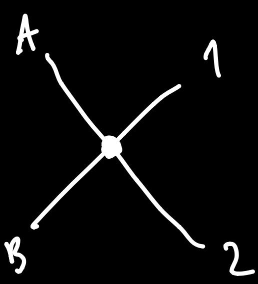
$\rightsquigarrow$ consider all possible full contractions of ϕ and external state momenta!

e.g. $\bullet\; -i\frac{\lambda}{4!}\int d^4x\; {}_0\langle \vec{p}_1 \vec{p}_2 \,|\, \phi\,\phi\,\phi\,\phi \,|\, \vec{k}_A \vec{k}_B\rangle_0$ $(\cdot\, 3\times 2)$

$= 8 \times$ 

part of the "1" in $S = 1 + iT$

$\rightsquigarrow$ ignore



- $-i\frac{\lambda}{4!}\int d^4x \; {}_0\langle \vec{p}_1\, \vec{p}_2 \,|\, \phi\,\phi\,\phi\,\phi \,|\, \vec{k}_A\, \vec{k}_B \rangle_0$ (4! options)

$$= 4!\left(-i\frac{\lambda}{4!}\right)\int d^4x \; e^{+ip_1x}\, e^{+ip_2x}\, e^{-ik_Ax}\, e^{-ik_Bx}\, \underbrace{\langle 0|0\rangle}_{1}$$

$$= \underbrace{-i\lambda}_{= i\mathcal{M}}\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-k_A-k_B)$$

✓ when directly applying Feynman rules!

7. Feynman rules for fermions

Wick's theorem

$$\boxed{T\{\psi_1\bar{\psi}_2\psi_3\dots\} = N\{\psi_1\bar{\psi}_2\psi_3\dots + \text{all possible contractions}\}}$$

where • $\overbrace{\psi(x)\bar{\psi}(y)} \equiv \begin{cases} \{\psi^+(x), \bar{\psi}^-(x)\} & \text{for } x^0 > y^0 \\ -\{\bar{\psi}^+(x), \psi^-(x)\} & \text{for } x^0 < y^0 \end{cases} = S_F(x-y)$

• $\overbrace{\psi\psi} = \overbrace{\bar{\psi}\bar{\psi}} = 0$

• $N(\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4) = -N(\psi_1\bar{\psi}_3\psi_2\bar{\psi}_4)$

$= -\overbrace{\psi_1\bar{\psi}_3}\, N(\psi_2\bar{\psi}_4)$

etc.

contractions with external states

e.g. $\psi^+(x)\,|\vec{p},s\rangle_{\text{fermion}} = \int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'} u^{s'}(p')\,a_{p'}^{s'}\,e^{-ip'x}\sqrt{2E_p}\,a_p^{s\dagger}|0\rangle$

$= e^{-ipx}\,u^s(p)\,|0\rangle$

$\equiv \overbrace{\psi_I(x)\,|\vec{p},s\rangle}_{\text{fermion}}$

similarly: $\overbrace{\bar{\psi}(x)\,|\vec{p},s\rangle}_{\text{anti-fermion}} = e^{-ipx}\,\bar{v}^s(p)\,|0\rangle$

Yukawa theory

$$H = H_{Dirac} + H_{Klein\text{-}Gordon} + H_I$$

$$H_I = g\int d^3x\, \bar{\psi}\psi\phi$$

<u>example</u>: consider fermion (k) + fermion (p) $\longrightarrow$ fermion (k') + fermion (p')

$\rightarrow$ leading contribution to S-matrix from H_I^2 term:

$${}_0\langle \vec{p}\,'\vec{k}\,' \,|\, T\{\tfrac{1}{2!}(-ig)^2\int d^4x\int d^4y\,(\bar{\psi}_a\psi_a\phi)_x\,(\bar{\psi}\psi\phi)_y\}\,|\,\vec{k}\,\vec{p}\rangle_0$$

$\rightsquigarrow$ 2x2 contractions possible

$$\supset (-ig)^2\int d^4x\int d^4y\,(\bar{u}_a(p')u_a(p))(\bar{u}(k')u(k))\times$$

$$\times\, e^{+ik'y}e^{+ip'x}e^{-iky}e^{-ipx}$$

$$\times\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\varepsilon}e^{-iq(x-y)}$$

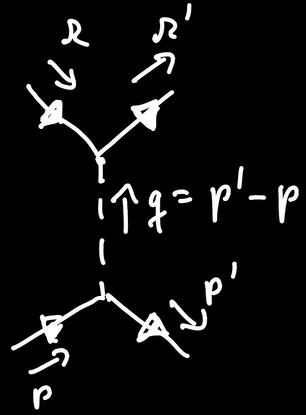
$$= -ig^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m_\phi^2+i\varepsilon}(2\pi)^8\,\delta^{(4)}(p'-p-q)\,\underbrace{\delta^{(4)}(k'-k+q)}_{\Rightarrow\,\delta^{(4)}(k'-k+p'-p)}$$

$\Rightarrow q = p'-p$

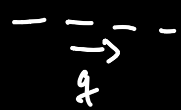
$$\times\,(\bar{u}_a(p')u_a(p))(\bar{u}(k')u(k))$$

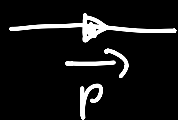
$$= \underbrace{\frac{-ig^2}{(p'-p)^2 - m_\phi^2} \left(\bar u_a(p')\, u_a(p)\right) \left(\bar u(k')\, u(k)\right)}_{= i\mathcal{M}} \cdot (2\pi)^4\, \delta^{(4)}(p+k-p'-k')$$

$\Leftarrow$ with Feynman rules:

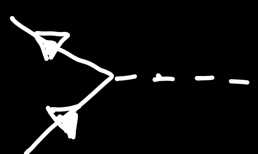


1. propagators

$\phi\,\phi$  $= \dfrac{i}{q^2 - m_\phi^2 + i\varepsilon}$

$\psi\,\bar\psi$  $= \dfrac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}$

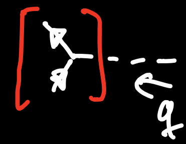
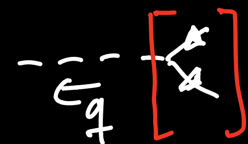
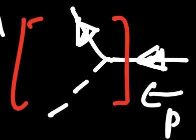
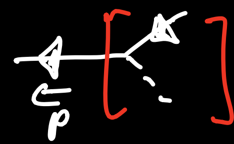
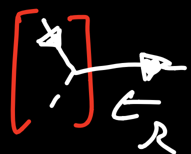
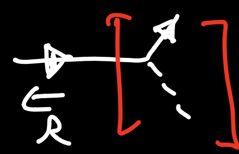
2. vertices

 $= -ig$

& impose 4-momentum conservation @ every vertex

& integrate over all undetermined (loop) momenta ⌈keep "$+i\varepsilon$" only for this case!⌋

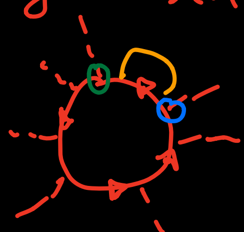
3. external leg contractions:

ingoing particles	outgoing particles
$\phi\,|\vec q\rangle \;\hat{=}\;$  $= 1$	$\langle \vec q\,|\,\phi^{+} \;\hat{=}\;$  $= 1$
$\psi\,|\vec p, s\rangle_{\text{fermion}} \;\hat{=}\;$  $= u^s(p)$	$\langle \vec p, s\,|\,\bar\psi \;\hat{=}\;$  $= \bar u^s(p)$
$\bar\psi\,|\vec k, s\rangle_{\text{anti-fermion}} \;\hat{=}\;$  $= \bar v^s(k)$	$\langle \vec k, s\,|\,\psi \;\hat{=}\;$  $= v^s(p)$

4. overall sign?

→ determined by anti-commutation of ψ!

e.g. closed fermion loop:



$$= (\bar\psi_a \psi_a)_x (\bar\psi_b \psi_b)_y \dots (\bar\psi_d \psi_d)$$

$$= -\psi_d \bar\psi_a \;\psi_a \bar\psi_b \;\dots\; \psi_b \bar\psi_d$$

$$= -\; S_{da} S_{ab} \dots S_{bd} = (-)\,\mathrm{Tr}[S \dots S]$$

⇝ 1 overall minus sign per fermion loop

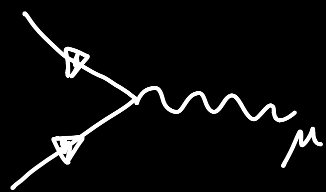
5. <u>no</u> symmetry factors!

because H_I has 3 <u>different</u> term

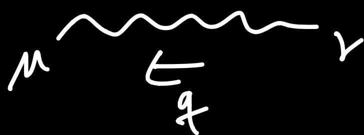
Quantum Electrodynamics (QED)

$$H_{int} = \int d^3x \; g\,\bar\psi\psi\phi \;\to\; \int d^3x \; e\,\bar\psi\gamma^\mu\psi A_\mu = \int d^3x \; e\,\bar\psi \not{A}\psi$$

=>
「proof later」

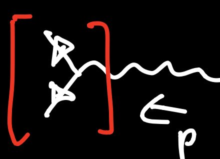


$= -ie\gamma^\mu \quad (= -iQ|e|\gamma^\mu)$



$$= \frac{-ig^{\mu\nu}}{q^2 + i\varepsilon}$$

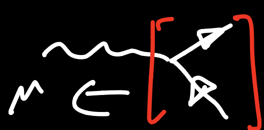
$A_\mu |\vec p\rangle \;\hat{=}$



$\mu = \varepsilon_\mu(p)$ "polarization vector"

w/ $\vec p \cdot \vec\varepsilon = 0$

$\langle \vec p | A_\mu \;\hat{=}$



$= \varepsilon^*_\mu(p)$

$p\cdot\varepsilon = 0$
from e.o.m., only for massless particles

e.g. $\varepsilon^\pm = (0, 1, \pm i, 0)\cdot\frac{1}{\sqrt{2}}$
for circular polarization

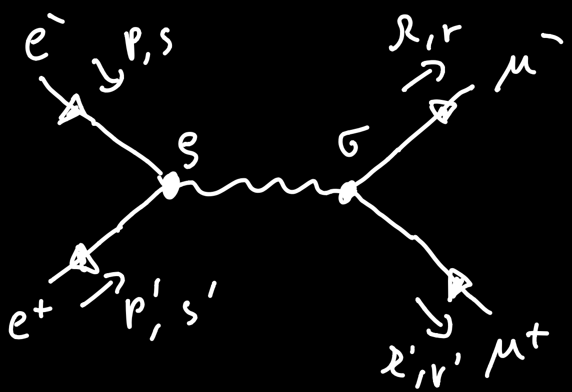
8. Elementary QED processes

1st example: muon production $e^+e^- \to \mu^+\mu^-$

μ : "heavy electron" ($m_\mu \simeq 106$ MeV $\simeq 200\, m_e$)

$\leadsto$ same Feynman rules

$\Rightarrow$ lowest order = "tree level" (i.e. no loops):



in $\xrightarrow[t]{}$ out

$\Rightarrow$ contributions to $i\mathcal{M}$:

1. Fermion chains (against arrow direction!)

   a) muons: $\bar{u}^r(k)\,(-ie\gamma^\sigma)\,v^{r'}(k')$

   b) electrons: $\bar{v}^{s'}(p')\,(-ie\gamma^\rho)\,u^s(p)$

2. photon propagator:

$$\frac{-i g_{\rho\sigma}}{(p+p')^2 \cancel{+i\varepsilon}} \equiv \frac{-i g_{\rho\sigma}}{s} \qquad \text{"s-channel"}$$

$$\Rightarrow i\mathcal{M} = \frac{ie^2}{s}\left(\bar{u}^r(k)\,\gamma^\sigma v^{r'}(k')\right)\left(\bar{v}^{s'}(p')\,\gamma_\sigma u^s(p)\right)$$

$$(\bar u \gamma^\rho v)^* = (u^\dagger \gamma^0 \gamma^\rho v)^\dagger = v^\dagger \underbrace{\gamma^{\rho\dagger}\gamma^{0\dagger}}_{\gamma^0\gamma^\rho} u = \bar v \gamma^\rho u$$

$$\Rightarrow |\mathcal{M}|^2 = \frac{e^4}{s^2}\left[(\bar u(k)\gamma^\sigma v(k'))(\bar v(p')\gamma_\sigma u(p))\right]$$
$$\times \left[(\bar v(k')\gamma^\rho u(k))(\bar u(p)\gamma_\rho v(p'))\right]$$

typically interested in unpolarized cross sections, i.e.

$$\boxed{|\mathcal{M}|^2 \longrightarrow \frac{1}{(2s_A+1)(2s_B+1)} \sum_{s,s',r,r'} |\mathcal{M}|^2 \equiv |\bar{\mathcal{M}}|^2}$$

"sum over final, average over initial spin states"

only for massive particles!

$\Leftrightarrow$ photon : 2 d.o.f.

$\rightarrow$ can use $\sum_s u^s(p)\bar u^s(p) = \not p + m$!

$$\Rightarrow \text{e.g. } \_\_\times\_\_ = \sum_{r,r'} \bar u^r_a(k)\gamma^\sigma_{ab} \underbrace{v^{r'}_b(k')\bar v^{r'}_c(k')}_{=(\not k' - m_\mu)_{bc}} \gamma^\rho_{cd} u^r_d(k)$$

$= (\not k + m_\mu)_{da}$

$$= \mathrm{Tr}[(\not{k} + m_\mu)\gamma^\sigma(\not{k}' - m_\mu)\gamma^\rho]$$

very generic expression that appears in calculating $|\bar{\mathcal{M}}|^2$

⇝ useful to collect properties of <u>traces</u> of γ matrices

- $\mathrm{Tr}[(\text{any odd \# of } \gamma\text{'s})] = 0$

$$\ulcorner = \mathrm{Tr}[\gamma^5\gamma^5(\;)] \quad | \quad \{\gamma^5, \gamma^\mu\} = 0$$

$$= -\mathrm{Tr}[\gamma^5(\;)\gamma^5] \quad | \quad \mathrm{Tr}[A_1 \cdot A_2 \cdot \ldots A_{n-1} \cdot A_n] = \mathrm{Tr}[A_n \cdot A_1 \cdot A_2 \cdots A_{n-1}]$$

$$= -\mathrm{Tr}[\underbrace{\gamma^5\gamma^5}_{1}(\;)] \quad \lrcorner$$

- $\mathrm{Tr}[\mathbb{1}] = 4$
- $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

$$\ulcorner = \mathrm{Tr}[2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu]$$

$$= 8g^{\mu\nu} - \underbrace{\mathrm{Tr}[\gamma^\nu\gamma^\mu]}_{\mathrm{Tr}[\gamma^\mu\gamma^\nu]} \quad \square \quad \lrcorner$$

- $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = [\ldots]$

$$= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

- $\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\ldots] = \mathrm{Tr}[\ldots\gamma^{\mu_2}\gamma^{\mu_3}]$

$$\Rightarrow \bullet\ \mathrm{Tr}[(\not{k} + m_\mu)\gamma^\sigma(\not{k}' - m_\mu)\gamma^\rho]$$

$$= \underbrace{\mathrm{Tr}[\not{k}\gamma^\sigma\not{k}'\gamma^\rho]}_{k_\mu k'_\nu \mathrm{Tr}[\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho]} - m_\mu^2 \underbrace{\mathrm{Tr}[\gamma^\sigma\gamma^\rho]}_{4g^{\sigma\rho}}$$

$$= 4[k^\sigma k'^\rho - (k\cdot k')g^{\sigma\rho} + k^\rho k'^\sigma - m_\mu^2 g^{\sigma\rho}]$$

$$\bullet\ \mathrm{Tr}[(\not{p}' - m_e)\gamma_\sigma(\not{p} + m_e)\gamma_\rho] = \underline{(k^{(\prime)} \to p^{(\prime)},\ m_\mu \to m_e)}$$

$$\Rightarrow |\bar{\mathcal{M}}|^2 = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4e^4}{s}[k^\rho k'^\sigma + k^\sigma k'^\rho - g^{\rho\sigma}(k\cdot k' - m_\mu)]$$

$$\times[p_\rho p'_\sigma + p_\sigma p'_\rho - g_{\rho\sigma}(p\cdot p' - m_e^2)]$$

$\downarrow$ $m_e \ll m_\mu$ $\quad$ [NB: $\alpha = \frac{e^2}{4\pi} \sim 10^{-2}$!]

$$\simeq \frac{8e^4}{s^2}[(k\cdot p)(k'\cdot p') + (k'\cdot p)(k\cdot p') + m_\mu^2\, p\cdot p']$$

now 2 options: a) choose a reference frame and evaluate contractions explicitly

$\rightsquigarrow$ angular dependence $\rightsquigarrow$ see P&S

b) keep everything Lorentz-invariant

@b) $\bullet\ k\cdot p = -\frac{1}{2}[\underbrace{(k-p)^2}_{t} - \underbrace{\cancel{m_e^2}}_{= k^2} - \underbrace{m_\mu^2}_{= p^2}] = k'\cdot p'$

$p + p' = k + k'$

$\bullet\ k\cdot p' = [\sim t \to u] = -\frac{1}{2}[-s - t + \cancel{m_e^2} + m_\mu^2] = k'\cdot p$

$\bullet \; p \cdot p' = \frac{1}{2}\left[\underbrace{(p+p')^2}_{s} - 2m_e^2\right]$

$[\cdot \; k \cdot k' = \frac{1}{2}\left[s - 2m_\mu^2\right]$

$$\Rightarrow \ldots \quad \boxed{\overline{|\mathcal{M}|^2} = \frac{4e^4}{s^2}\left[(t - m_\mu^2)^2 + \frac{s^2}{2} + st\right]}$$

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s}\,\frac{1}{\vec{p}_{cm}^{\,2}}\,\overline{|\mathcal{M}|^2}$$

cms: $\vec{p} = -\vec{p}\,'$ $\Rightarrow$ $E_p = E_{p'}$ ($m_e = m_e'$)

$\Rightarrow s = (p+p')^2 = 4E_{cm}^2 = 4E_p^2$

$\vec{p}_{cms}^{\,2} = E_p^2 - m_e^2 = \frac{s}{4} - m_e^4$

$k_{cms}^2 = \frac{s}{4} - m_\mu^2$

$\Rightarrow t_{min/max} = (|\vec{p}|_{cm} \mp |\vec{k}|_{cm})^2$

$\rightsquigarrow \quad = m_\mu^2 - \frac{s}{2}\left(1 \mp \sqrt{1 - \frac{4m_\mu^2}{s}}\right)$

phase-space suppression

$$\Rightarrow \boxed{\sigma \simeq \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{4m_\mu^2}{s}}\left(1 + \frac{2m_\mu^2}{s}\right)} \quad \text{where } \alpha = \frac{e^2}{4\pi}$$

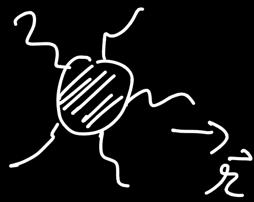
NB: $\bullet \; \sigma \sim \frac{\alpha^2}{s}$ ✓

$\bullet \; s \geq 4m_\mu^2 \iff E_{cm} \overset{!}{\geq} 2m_\mu$

for $\sigma > 0$ ✓

$\rightsquigarrow$ agreement with naive expectation! ✓

photon polarizations



$$\Rightarrow \quad i\mathcal{M} = i\,\mathcal{M}^\mu\, \varepsilon^*_\mu(k)$$

$\gamma\, A_\mu \to \varepsilon^*_\mu$

$$\mathcal{M}^\mu \leftarrow \int d^4x\, e^{ikx} \langle \text{final} \,|\, \underbrace{\bar\psi(x)\gamma^\mu\psi(x)}_{\equiv j^\mu} \,|\, \text{initial} \rangle$$

$$\Rightarrow k_\mu \mathcal{M}^\mu \propto \int d^4x \left(\partial_\mu e^{ikx}\right) \langle \text{final} \,|\, j^\mu \,|\, \text{initial} \rangle$$

$$= -\int d^4x\, e^{ikx} \langle \text{final} \,|\, \underbrace{\partial_\mu j^\mu}_{=0 \text{ (classical e.o.m)!}} \,|\, \text{initial} \rangle$$

$$\Rightarrow \boxed{k_\mu \mathcal{M}^\mu = 0} \quad (*)$$

"Ward identity"

[general proof: QFT; consequence of gauge invariance / current conservation]

$$\Rightarrow \sum_{\text{photon polarizations}} |\mathcal{M}|^2 = \sum_\varepsilon \varepsilon^*_\mu(k)\, \varepsilon_\nu(k)\, \mathcal{M}^\mu(k)\, \mathcal{M}^\nu(k)^*$$

choose $k^\mu = (k, 0, 0, k)$ + transverse polarizations: $\varepsilon_1^\mu = (0,1,0,0)$, $\varepsilon_2^\mu = (0,0,1,0)$

$$= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2$$

($\varepsilon = \varepsilon_1$, $\varepsilon = \varepsilon_2$)

(*) $\Rightarrow$ $k\mathcal{M}^0 - k\mathcal{M}^3 = 0$

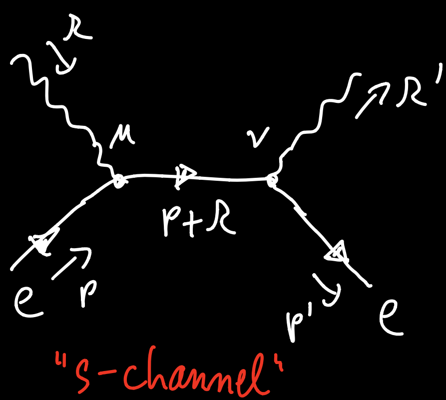
$$= -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2$$

$$= -g_{\mu\nu}\, \mathcal{M}^\mu \mathcal{M}^{\nu *}$$

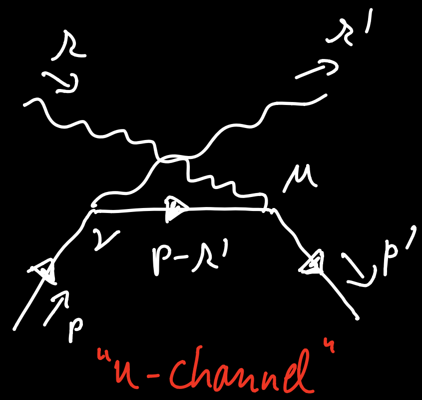
i.e. $$\boxed{\sum_\varepsilon \varepsilon^*_\mu \varepsilon_\nu \longrightarrow -g_{\mu\nu}}$$

NB: not an equality!

2nd example : Compton scattering, $e^-\gamma \to e^-\gamma$



$+$



NB : fermion parts identical
$\Rightarrow$ no relative minus sign !

$$\Rightarrow i\mathcal{M} = (-ie)^2\, \bar{u}(p') \left\{ \frac{\gamma^\nu i(\not{p}+\not{k}+m)\gamma^\mu}{\underbrace{(p+k)^2}_{s}-m^2} + \frac{\gamma^\mu\, i(\not{p}-\not{k}'+m)\gamma^\nu}{\underbrace{(p-k')^2}_{u}-m^2} \right\} u(p)\, \varepsilon^*_\nu(k')\, \varepsilon_\mu(k)$$

use Dirac equation to simplify !

$$(\not{p}+m)\gamma^\mu u(p) = 2p^\mu u(p) + \gamma^\mu \underbrace{(-\not{p}+m)u(p)}_{=0}$$

$$\Rightarrow i\mathcal{M} = -ie^2\, \bar{u}(p') \left\{ \frac{\gamma^\nu \not{k}\gamma^\mu + 2\gamma^\nu p^\mu}{s-m^2} + \frac{-\gamma^\mu \not{k}'\gamma^\nu + 2\gamma^\mu p^\nu}{t-m^2} \right\} u(p)\,\varepsilon_\mu(k)\varepsilon^*_\nu(k')$$

$$\Rightarrow |\bar{\mathcal{M}}|^2 = \frac{1}{\underset{e^-}{2}\cdot\underset{\gamma}{2}} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4} \sum_{\text{photon spins}} \underbrace{\varepsilon_\mu(k)\,\varepsilon^*_{\mu'}(k)}_{\to\, g_{\mu\mu'}}\ \underbrace{\varepsilon^*_\nu(k')\,\varepsilon_{\nu'}(k')}_{\to\, g_{\nu\nu'}}$$

$$\times \mathrm{Tr}\Big[(\not p' + m)\Big\{\frac{\gamma^\nu \not k \gamma^\mu + 2\gamma^\nu p^\mu}{s-m^2} + \frac{-\gamma^\mu \not k' \gamma^\nu + 2\gamma^\mu p^\nu}{u-m^2}\Big\}$$

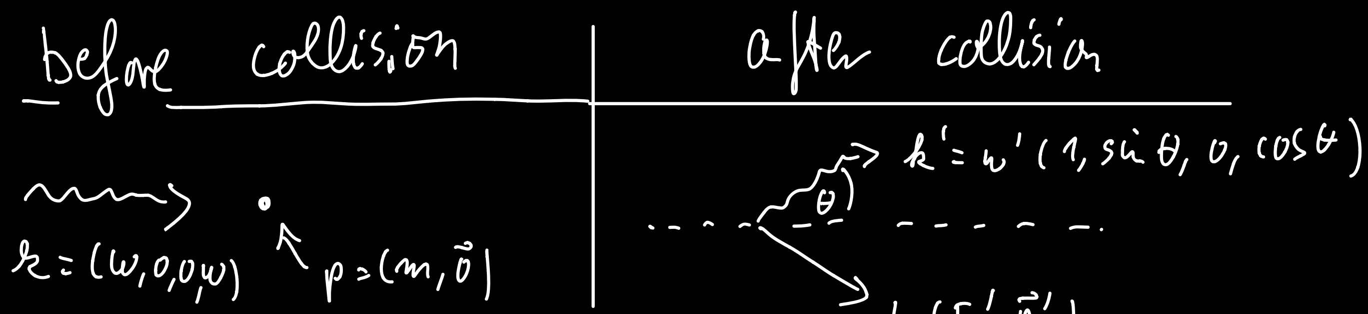
$$\times (\not p + m)\Big\{\frac{\gamma^{\mu'} \not k \gamma^{\nu'} + 2\gamma^{\nu'} p^{\mu'}}{s-m^2} + \frac{-\gamma^{\nu'} \not k' \gamma^{\mu'} + 2\gamma^{\mu'} p^{\nu'}}{u-m^2}\Big\}\Big]$$

$$(\bar\psi_1 \gamma^\mu \gamma^\nu \psi_2)^\dagger = \bar\psi_2 \gamma^\nu \gamma^\mu \psi_1$$

- identify symmetries between 4 terms from expanding the sum:
  - cross terms $\propto \frac{1}{(s-m^2)}\frac{1}{(t-m^2)}$ identical
  - term $\propto \frac{1}{(u-m^2)^2}$ from $\frac{1}{(s-m^2)^2}$ term and $k \to -k'$
- $\not p \not p = p^2 = m^2$
- $\gamma^\mu \not p \gamma_\mu = -2\not p$
- ...

$$\rightsquigarrow \boxed{\overline{|\mathcal{M}|^2} = 2e^4\Big[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + \Big(1 + \frac{m^2}{p\cdot k} - \frac{m^2}{p\cdot k'}\Big)^2 - 1\Big]}$$

typically described in "lab" frame:



$$\Rightarrow p\cdot k' = m\omega'$$

$$p\cdot k = m\omega$$

- $\omega' = \omega'(\omega, \theta)$: $\cancel{m}^2 = p'^2 = (p + k - k')^2$

$$= p^2 + 2p\cdot(k-k') + \underbrace{(k-k')^2}_{-2k\cdot k'}$$

$$= \cancel{m}^2 + \cancel{2}m(\omega-\omega') - \cancel{2}\omega\omega'(1-\cos\theta)$$

$$\Rightarrow \omega'(m + \omega(1-\cos\theta)) = m\omega$$

$$\Leftrightarrow \boxed{\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\theta)}} \quad (*)$$

- $$\int d\Pi_2 = \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2\omega' 2E'} (2\pi)^4 \delta^{(4)}(k + p - k' - p')$$

$$= \int \frac{\omega'^2 d\Omega\, d\omega'}{(2\pi)^3} \frac{1}{4\omega' E'} 2\pi\, \delta(\omega + \underset{m}{\underset{\parallel}{E}} - \omega' - E') \quad E' = \sqrt{m^2 + (\vec{k} - \vec{k}' + \underset{0}{\underset{\parallel}{\vec{p}}})^2}$$

$$= \sqrt{m^2 + \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta}$$

$$\left| \int f(x)\delta(g(x))\,dx = \frac{1}{|g'(x_0)|} f(x_0) \right.$$

$$= \int \frac{d\cos\theta}{2\pi} \frac{\omega'}{4E'} \left| +1 + \frac{\cancel{2}\omega' - \cancel{2}\omega\cos\theta}{\cancel{2}E'} \right|^{-1}$$

$$= \int \frac{d\cos\theta}{8\pi} \frac{\omega'}{\underbrace{E' + \omega'}_{=E+\omega = m+\omega} - \omega\cos\theta} = \int \frac{d\cos\theta}{8\pi} \frac{\omega'}{m + \omega(1-\cos\theta)}$$

$$\overset{(*)}{=} \int \frac{d\cos\theta}{8\pi} \frac{(\omega')^2}{m\omega}$$

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{1}{2\underset{\overset{\parallel}{m}}{E}\, 2\omega\, \underbrace{|v_A - v_B|}_{1}} \; \frac{1}{8\pi} \frac{|\omega'|^2}{m\omega} \, |\overline{\mathcal{M}}|^2$$

$$= \frac{e^4}{16\pi m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + \left(1 + \frac{m}{\omega} - \frac{m}{\omega'}\right)^2 - 1 \right]$$

$$\left| \; \frac{m}{\omega} - \frac{m}{\omega'} = \frac{m}{\omega}\left(1 - \frac{\omega}{\omega'}\right) \right.$$

$$\overset{(*)}{=} \frac{m}{\omega}\left[1 - 1 - \frac{\omega}{m}(1 - \cos\theta)\right]$$

$$= -(1 - \cos\theta)$$

$$\Rightarrow \boxed{\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]}}$$

"Klein-Nishima" formula

$\omega \ll m$

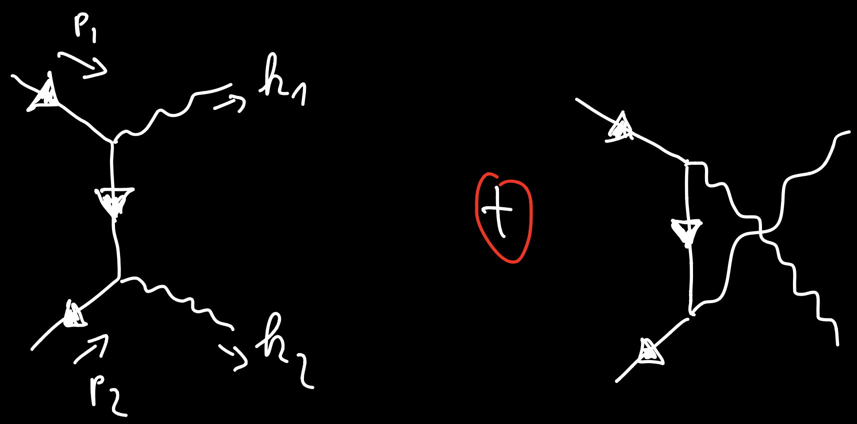
$\Rightarrow \omega = \omega'$

$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(1 + \cos^2\theta\right)}$$

Thompson cross section (classical EM!)

✓

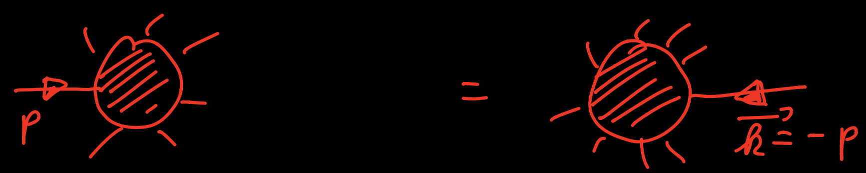
3^{rd} example – pair annihilation into photons: $e^+e^- \to \gamma\gamma$



⇝ this is related to Compton scattering by "crossing symmetry"

antiparticle of ϕ

general: $\mathcal{M}\,(\phi(p) + \dots \longrightarrow \dots) = \mathcal{M}\,(\dots \longrightarrow \dots + \bar{\phi}(-p))$



($p + \Sigma$ all other momenta $= 0$)

(Σ all other momenta $- k = 0$)

only difference in external legs:

$$\sum u(p)\,\bar{u}(p) = \not{p} + m = -(\not{k} - m)$$

$$= (-) \sum v(k)\,\bar{v}(k)$$

for each crossed fermion!

here : (compared to notation from Compton scattering case)

$$p \rightarrow p_1$$
$$p' \rightarrow -p_2$$
$$k' \rightarrow k_2$$
$$k \rightarrow -k_1$$
$$+\ \overline{|\mathcal{M}|^2} \rightarrow -\overline{|\mathcal{M}|^2}$$

$$\Rightarrow \overline{|\mathcal{M}|^2} = -2e^4\left[-\frac{p_1\cdot k_2}{p_1\cdot k_1} - \frac{p_1\cdot k_1}{p_1\cdot k_2} + \left(1-\frac{m^2}{p_1\cdot k_1}-\frac{m^2}{p_1\cdot k_2}\right)^2 - 1\right]$$

high-energy limit : $s \gg 4m^2$

$\Rightarrow$ CMS : $p_1 = (E, \vec{p})$; $p_2 = (E, -\vec{p})$; $E = |\vec{p}|$

$k_1 = (\omega, \vec{k})$; $k_2 = (\omega, -\vec{k})$; $\omega = |\vec{k}|$

$\Rightarrow p_1\cdot k_2 = E\omega\,(1+\cos\theta)$

$p_1\cdot k_1 = E\omega\,(1-\cos\theta)$

$$\Rightarrow \overline{|\mathcal{M}|^2} = 2e^4\left[\underbrace{\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta}}_{\frac{2+2\cos^2\theta}{\sin^2\theta}} + 0\right]$$

$m \ll E$!

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{1}{\underbrace{2E_A\,2E_B}_{=E_{cm}^2 = s}\underbrace{|v_A - v_B|}_{2}}\ \underbrace{\frac{|\vec{k}_1|}{8\pi E_{cm,2}}}_{}\ 4e^4\,\frac{1+\cos^2\theta}{\sin^2\theta}$$

$d\Pi_2/d\cos\theta$ in CMS

$$\Rightarrow \boxed{\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \frac{1+\cos^2\theta}{\sin^2\theta}}$$