Basics of group theory

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1 Introduction

The concepts of groups and their representations are vital in high-energy physics. Physical theories are often constructed from postulating invariance of the theory under certain transformations; each such class of transformations (usually indexed by some set of parameters) describes a group, with the specific transformations corresponding to elements of the group.

Examples of such transformations, which can be categorized in groups, include the Poincaré group — the group describing Lorentz transformations and translations in Minkowski space — and the matrix groups describing gauge transformations, such as the special unitary groups $\mathrm{SU}(N)$.

This note is intended to give a brief introduction to the group theory necessary for this course, by providing definitions and examples of groups and their representations. We will in particular focus on a class of groups called Lie groups; the representations of these groups can be constructed continuously from corresponding objects called Lie algebras.

2 Lie groups

First, we define exactly what is meant by a group.

A *group* G is a set $G = \{g\}$ along with an operation $\circ$, defined so that

- For any $g_1, g_2 \in G$,
$$g_1 \circ g_2 \in G. \tag{1}$$
- For any $g_1, g_2, g_3 \in G$,
$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3. \tag{2}$$
- There exists an identity element $I \in G$, such that for any $g \in G$,
$$I \circ g = g \circ I = g. \tag{3}$$
- Each element $g \in G$ has an inverse $g^{-1} \in G$, which satisfies
$$g \circ g^{-1} = g^{-1} \circ g = I. \tag{4}$$

A simple example of a group is the general linear group GL(N) of invertible $N \times N$ matrices, with the operation $\circ$ representing ordinary matrix multiplication. It is easy to demonstrate that these satisfy the definition of a group: (1) is satisfied since a matrix is invertible if and only if its determinant is different from zero, and since for any two matrices A and B, $\det(AB) = \det(A)\det(B)$; (2) follows from the associativity of matrix multiplication; (3) is satisfied by the identity matrix $I_{N\times N}$; and (4) is immediately satisfied by the previous requirement that the matrices be invertible.

The general linear group has several *subgroups* — groups that by themselves satisfy the group definition, but where each group element is also an element of some larger group with the same operation $\circ$. An example is the special unitary group SU(N), consisting of all unitary $N \times N$ matrices with determinants equal to 1. Demonstrating that these matrices, still using matrix multiplication as the operation $\circ$, satisfy the conditions (1)–(4) is left as an exercise.

In physics, groups are typically used to categorize various transformations. We usually want to parametrize such transformation groups in an analytic and continuous way; this is satisfied by the class of groups called *Lie groups.* Mathematically, these groups are geometrical objects called manifolds, which in practice means that they can be parametrized smoothly (by any number of parameters), and that the group operation $\circ$ is a smooth function of these parameters.

The previously discussed general linear group is a trivial example of a Lie group; since the elements of the matrices are just complex numbers, we can simply assign two parameters for each matrix element,[1] making matrix multiplication a simple analytical function of said parameters.

3 Representations

As we just noted, groups in physics are typically used to categorize transformations. These transformations must obviously act on something; this is typically some kind of vector, either a coordinate vector like the Minkowski coordinates x^μ, or for example a wave function. In general, these vectors belong to a vector space V.

In order for the transformation group elements to act on vectors in this space, we need some mapping of the group elements to the space of linear, invertible maps on V — specifically, we need a map $\rho : G \to \mathrm{GL}(V)$ where GL(V) is the subspace of GL(N) that maps V onto itself.[2] Such a map is called a *representation* of the group G; we denote its action by $\rho(g)v_i$, where $g \in G$ and $v_i \in V$, and require that it behaves as $\rho(g_1 \circ g_2) = \rho(g_1)\rho(g_2)$.

[1] Note that since the matrices are restricted to be invertible there is some redundancy in this parametrization, but this is unimportant for illustration purposes.

[2] N is here the dimension of the vector space V; if N is finite, the matrix representation as presented here is possible. If not — for example if V is a Hilbert space containing wave functions, which formally is infinite-dimensional — such a matrix representation cannot be explicitly constructed. We must then consider GL(V) to be the group of linear, invertible operators on V. The rest of the analysis works out in the same way, except that we cannot write down the representations quite as explicitly.

For the matrix groups like $\mathrm{SU}(N)$ these representations are often trivial, since by the definition of the groups their elements already belong to $\mathrm{GL}(N)$. The representation obtained directly from the definition of a matrix group is called its *fundamental representation.*

Perhaps the most important group for this course is the Poincaré group; in particular, a representation of this group based on the so-called Dirac matrices is a possible starting point for constructing the quantum theory of fermion fields. The Poincaré group describes Lorentz transformations, i.e. rotations and boosts, along with translations in four-vector space. In general, the representations of the Poincaré group are written as operators $U(\Lambda, a)$; here Λ describes the familiar Lorentz transformations, while a is a four-vector parametrizing the translation. You will explore some of these representations and their properties in detail during this course.

3.1 Irreducible representations

In physics, we are usually most interested in representations that are what we call *irreducible* representations, often abbreviated *irreps.* A representation is said to be irreducible if there is no subspace of V that is closed under its operation; that is, if there is no $W \subset V$ such that $\rho(g)w_i \in W$ for all $g \in G$, $w_i \in W$.

This is best illustrated by an example of the opposite, a *reducible* representation. Suppose that we want the representation of a group G to act on $\mathbb{R}^2$, and that we have a representation parametrized as

$$\rho(g) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \tag{5}$$

where $g \in G$ and $a, b, c \in \mathbb{R}$. From this we can see that the subspace of $\mathbb{R}^2$ spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is closed under the operation of the group representation, since $\rho(g)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ for all g.

The usefulness of irreps is partly due to a result called Schur's lemma,[3][4] which states that if any operator commutes with all elements of an irreducible representation, it is proportional to the identity, with the proportionality constant denoted by λ.

What this means in practice is that the states on which the elements of the irrep act, i.e. vectors in V, are eigenstates of this operator with eigenvalue λ. Such operators are called *Casimir operators*, and the *Casimir invariants* λ can be used to label the representation.

[3] I. Schur, Neue Begründung der Theorie der Gruppencharaktere, Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin (1905).

[4] The remainder of this section is not strictly necessary for the course, so there is no need for concern if anything is unclear here.

States with different physical properties often transform under different representations of groups; for example, as you will see in this course, fermions transform under a certain representation of the Poincaré group defined by the Dirac matrices. The Casimir operators of this group label representations according to their squared mass m^2 and spin quantum number s; the aforementioned fermion representation can be shown to satisfy $s = \frac{1}{2}$.

4 Lie algebras

We now return to the subject of Lie groups, and how we can construct representations for this type of groups. As noted above, Lie groups can be parametrized in a smooth, continuous way; this means that the action of their representation on states should similarly be continuous. We can therefore decompose the action of any element in a Lie group into a product of infinitesimal transformations. The group element giving such an infinitesimal transformation can be written as $I + ia_j g_j, j = 1, 2, \ldots, d$; here d is the dimension of the group (considering G as a vector space, d is the number of basis vectors), a_j are arbitrary infinitesimal coefficients (the chosen factorization, extracting a factor i, is conventional), and g_j are called the *generators* of the representation. The latter constitute a basis for the space of all such infinitesimal transformations. We can then write any arbitrary, finite, member g of the representation as

$$g = \lim_{n \to \infty} \left(I + \frac{1}{n} ia_j g_j \right)^n \tag{6}$$

$$\equiv \exp(ia_j g_j), \tag{7}$$

which is called the *exponential map* from the space of the generators to the representation space. a_j are now *finite* parameters.

The generators of a representation of G form a vector space, which we denote by $\mathfrak{g}$. Along with the commutation relation

$$[g_i, g_j] = if_{ijk} g_k, \tag{8}$$

they form what is called the *Lie algebra* of the Lie group. The constants f_{ijk}, where $i, j, k = 1, 2, \ldots, d$, are called the *structure constants* of the algebra. In practice, we typically refer to (8) as the algebra of a Lie group, and let any set of matrices that fulfill the same algebra form a representation of the group through the exponential map (7).

For completeness, the general definition of a Lie algebra is as follows:

A *Lie algebra* is a linear vector space V defined along with a binary operator[5] $[,] : V \times V \to V$, that satisfies the following conditions:

- For any $v_i, w_i \in V$, $x_i, y_i \in \mathbb{C}$,

$$[x_i v_i, y_j w_j] = x_i y_j [v_i, w_j]. \tag{9}$$

- For any $v, w \in V$,
$$[v, w] = -[w, v]. \tag{10}$$
- For any $v_1, v_2, v_3 \in V$,
$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_2, v_2]] = 0, \tag{11}$$
which is called the Jacobi identity. Writing the Lie bracket for a set of basis vectors as in (8), the Jacobi identity can be expressed in terms of the structure constants as
$$f_{ijm} f_{mkl} + f_{jkm} f_{mil} + f_{kim} f_{mjl} = 0. \tag{12}$$

Since all elements of a representation can be constructed from the generators through the exponential map, we usually just specify a representation by its generators. These describe infinitesimal transformations where we can expand around some small parameter, and are often easier to work with than the "full" group members. An important example, that appears frequently in gauge theories, is the *adjoint representation;* the generators g_i^A of this representation are defined through the structure constants as
$$\left(g_i^A\right)_{jk} = -i f_{ijk}. \tag{13}$$
It can be shown through the Jacobi identity (12) that the generators of the adjoint representation indeed satisfy the algebra (8).

This means that for matrix groups, the dimension of the adjoint representation (really the dimension of the vector space on which it acts) will in general differ from that of the fundamental representation; the fundamental representation of GL(N) clearly has dimension N, whereas the adjoint representation has dimension equal to the number of generators. As a side note, this is why in Quantum Chromodynamics, the theory of strong interactions in the Standard Model, quarks appear in triplets while gluons appear in octets, since they transform in the fundamental and adjoint representations, respectively, of SU(3).

[5] This operator is called a *Lie bracket;* as the notation suggests it is often the commutator $[A, B] = AB - BA$, where the Lie algebra definition is automatically satisfied, but any operator that satisfies the same conditions works.