University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 10

Problem 1 Contractions for fermion fields (J. Skaar)

a) Wick's theorem for two fermion fields is

$$T\psi(x)\overline{\psi}(y) = N\psi(x)\overline{\psi}(y) + \contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y). \tag{1}$$

If (1) is taken as the definition of the contraction $\contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y)$, and

$$S_F(x-y) = \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle, \tag{2}$$

prove that

$$\begin{aligned}
\contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y) &= \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle = S_F(x-y) \\
&= \begin{cases} \{\psi^+(x), \overline{\psi}^-(y)\}, & x^0 > y^0, \\ -\{\overline{\psi}^+(y), \psi^-(x)\}, & x^0 < y^0, \end{cases} \\
&= -\contraction{}{\overline{\psi}}{(y)}{\psi}\overline{\psi}(y)\psi(x),
\end{aligned} \tag{3}$$

and

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = 0 = \contraction{}{\overline{\psi}}{(x)}{\overline{\psi}}\overline{\psi}(x)\overline{\psi}(y) \tag{4}$$

(Strictly speaking, we need spinor indices on the spinors above to make sense of the equations).

b) For all examples in P&S (see e.g. the section about Yukawa theory in Ch. 4), none of them has contractions between two external states (between two initial-state particles, between two final-state particles, or between initial-state and final-state particles). Can you explain why this possibility is ignored?

Problem 2 Interaction between fermions and a classical electromagnetic field

a) Peskin & Schroeder problem 4.4a p. 129. Note that your result can be represented as a Feynman diagram with associated rule (see 4.4b).

b) Specialize your result to the case where the classical field is a scalar potential $V(\mathbf{x})$ which is independent of time. Also assume that the particle is nonrelativistic.

> **Hint:** You need the equation below (4.133) in P&S.

c) In what sense is 4-momentum conserved in this process, in the general case and in the special case where the classical field is independent of time?

Problem 3 Spinor algebra and trace methods (T. Klungland)

a) For any operator Γ consisting of a product of an arbitrary number of γ matrices, and any two spinors $v^s(p)$, $u^r(k)$ (r and s label the spin states of the two spinors) (whether the spinors are particle or antiparticle spinors is irrelevant; these are just chosen as an example), show that

$$\left(\overline{v}^s(p)\Gamma u^r(k)\right)^\dagger = \overline{u}^r(k)\Gamma' v^s(p), \tag{5}$$

where $\overline{u} = u^\dagger\gamma^0$ and Γ' is the same product as Γ with the order of matrices reversed. You will need the identity $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

b) Suppose that the scattering of a fermion off of some potential is described by the matrix element

$$i\mathcal{M} = iB_\mu \overline{u}^r\left(p'\right)\gamma^\mu u^s(p),$$

where p and p' describe the initial and final momenta, respectively, and s and r the initial and final spins. B_μ contains the other numerical factors, that are unimportant for our purposes. Use the spin sum relation in Eq. (3.66) in P&S to show that the unpolarized, spin-averaged squared matrix element (this means that we average over the 2 possible initial-state spins and sum over the final-state ones) is given by

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{2}\sum_{s=1}^{2}\sum_{r=1}^{2}\left[B_\mu \overline{u}^r\left(p'\right)\gamma^\mu u^s(p)\right]^\dagger \left[B_\nu \overline{u}^r\left(p'\right)\gamma^\nu u^s(p)\right] \tag{6}$$

$$= \frac{1}{2}B_\mu^* B_\nu \mathrm{Tr}\left[\left(\not{p}' + m\right)\gamma^\mu(\not{p} + m)\gamma^\nu\right], \tag{7}$$

where m is the mass of the fermion and $\not{p} = p_\mu\gamma^\mu$.

Hint: One way of doing this is explained on p. 132 in P&S, but it is easier to use a) that a product of matrices that gives a scalar, or a 1×1 matrix can be written as its own trace, and b) the cyclic and linear properties of traces.