# University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 10

## Problem 1 Contractions for fermion fields (J. Skaar)

**a)** Wick's theorem for two fermion fields is

$$T\psi(x)\overline{\psi}(y) = N\psi(x)\overline{\psi}(y) + \overbracket{\psi(x)\overline{\psi}}(y). \tag{1}$$

If (1) is taken as the definition of the contraction $\overbracket{\psi(x)\overline{\psi}}(y)$, and

$$S_F(x-y) = \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle, \tag{2}$$

prove that

$$\begin{aligned} \overbracket{\psi(x)\overline{\psi}}(y) &= \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle = S_F(x-y) \\ &= \begin{cases} \{\psi^+(x), \overline{\psi}^-(y)\}, & x^0 > y^0, \\ -\{\overline{\psi}^+(y), \psi^-(x)\}, & x^0 < y^0, \end{cases} \\ &= -\overbracket{\overline{\psi}(y)\psi}(x), \end{aligned} \tag{3}$$

and

$$\overbracket{\psi(x)\psi}(y) = 0 = \overbracket{\overline{\psi}(x)\overline{\psi}}(y) \tag{4}$$

(Strictly speaking, we need spinor indices on the spinors above to make sense of the equations).

**Solution:** The first equality follows by taking the vacuum expectation value of (1), since the vacuum expectation value of normal ordered operators is zero. For the last equality, note that for $x^0 > y^0$,

$$\begin{aligned} T\psi(x)\overline{\psi}(y) = \psi(x)\overline{\psi}(y) &= \psi^+(x)\overline{\psi}^+(y) + \psi^+(x)\overline{\psi}^-(y) \\ &\quad + \psi^-(x)\overline{\psi}^+(y) + \psi^-(x)\overline{\psi}^-(y). \end{aligned} \tag{5}$$

Taking the vacuum expectation value (and taking into account the usual expressions for $\psi(x)$ and $\overline{\psi}(y)$ in terms of ladder operators), we obtain

$$\langle 0|T\psi(x)\overline{\psi}(y)|0\rangle = \langle 0|\psi^+(x)\overline{\psi}^-(y)|0\rangle = \{\psi^+(x), \overline{\psi}^-(y)\}. \tag{6}$$

When $x^0 < y^0$ we have $T\psi(x)\overline{\psi}(y) = -\overline{\psi}(y)\psi(x)$, and proceed in the same way.

**b)** For all examples in P&S (see e.g. the section about Yukawa theory in Ch. 4), none of them has contractions between two external states (between two initial-state particles, between two final-state particles, or between initial-state and final-state particles). Can you explain why this possibility is ignored?

## Problem 2 Interaction between fermions and a classical electromagnetic field

**a)** Peskin & Schroeder problem 4.4a p. 129. Note that your result can be represented as a Feynman diagram with associated rule (see 4.4b).

**Solution:** Write $S = 1 + iT$ (and always keep the $i$ in $iT$ to make it different from the time ordering symbol $T$):

$$\begin{aligned}\langle p'|iT|p\rangle &= {}_0\langle p'|Te^{-i\int dt H_I(t)}|p\rangle_{0,\text{connected, amputated}} \\ &= {}_0\langle p'|Te^{-i\int d^4x e\overline{\psi}\gamma^\mu\psi A_\mu}|p\rangle_{0,\text{connected, amputated}}. \end{aligned} \tag{7}$$

Here we have used

$$H_I(t) = \int d^3x e\overline{\psi}\gamma^\mu\psi A_\mu, \tag{8}$$

where $A_\mu$ represent the classical, electromagnetic field, and $\psi$ is the interaction-picture (quantized) Dirac field.

To first order:

$$\begin{aligned}\langle p'|iT|p\rangle &= -ie\int d^4x A_\mu \cdot {}_0\langle p'|T\overline{\psi}\gamma^\mu\psi|p\rangle_0 = -ie\int d^4x A_\mu \cdot {}_0\langle p'|\overline{\psi}\gamma^\mu\psi|p\rangle_0 \\ &= -ie\int d^4x A_\mu(x)e^{i(p'-p)x}\overline{u}(p')\gamma^\mu u(p) \\ &= -ie\tilde{A}_\mu(p'-p)\overline{u}(p')\gamma^\mu u(p). \end{aligned} \tag{9}$$

To obtain the second line, we use that the contraction between $\psi(x)$ and $|p\rangle_0$ gives $u(p)e^{-ipx}$. The other contraction gives $\overline{u}(p')e^{ip'x}$. Note the presence of the exponentials. (In the momentum-space Feynman rules for QED or Yukawa theory, the exponentials are omitted, but that is because they haved been used together with $\int d^4x$ to obtain momentum-conserving delta functions.)

We note that (9) can be represented by the Feynman diagram and Feynman rule mentioned in 4.4b.

**b)** Specialize your result to the case where the classical field is a scalar potential $V(\mathbf{x})$ which is independent of time. Also assume that the particle is nonrelativistic.

**Hint:** You need the equation below (4.133) in P&S.

**Solution:** To get a nonzero result, the input and output spins must match. The result is

$$\langle p'|iT|p\rangle = -i\tilde{V}(\mathbf{q})\, 2\pi\delta(E_{\mathbf{p}'} - E_{\mathbf{p}}) \cdot 2m, \quad \mathbf{q} = \mathbf{p}' - \mathbf{p}, \tag{10}$$

which is (4.124) in P&S, except for the extra $2m$ factor due to the relativistic normalization of $|p\rangle$ and $|p'\rangle$ here.

**c)** In what sense is 4-momentum conserved in this process, in the general case and in the special case where the classical field is independent of time?

**Solution:** We have $p' = p + q$, i.e., the output momentum is equal to the input momentum plus the momentum from the Fourier component of the classical field. In the general case the classical field consists of a spectrum of Fourier components with momenta $q$; this leads to a spectrum of output momenta $p'$ even though $p$ is fixed. When the classical field is independent of time the Fourier transform contains a delta function $\delta(q^0)$ so the input and output energies of the fermion are equal.

## Problem 3 Spinor algebra and trace methods (T. Klungland)

**a)** For any operator $\Gamma$ consisting of a product of an arbitrary number of $\gamma$ matrices, and any two spinors $v^s(p)$, $u^r(k)$ ($r$ and $s$ label the spin states of the two spinors) (whether the spinors are particle or antiparticle spinors is irrelevant; these are just chosen as an example), show that

$$(\overline{v}^s(p)\Gamma u^r(k))^\dagger = \overline{u}^r(k)\Gamma' v^s(p), \tag{11}$$

where $\overline{u} = u^\dagger\gamma^0$ and $\Gamma'$ is the same product as $\Gamma$ with the order of matrices reversed. You will need the identity $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

**Solution:** Writing $\Gamma$ as $\Gamma = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}$, its hermitian conjugate is given by

$$\begin{aligned}\Gamma^\dagger &= (\gamma^{\mu_n})^\dagger\cdots(\gamma^{\mu_2})^\dagger(\gamma^{\mu_1})^\dagger \\ &= \gamma^0\gamma^{\mu_n}\gamma^0\cdots\gamma^0\gamma^{\mu_2}\gamma^0\gamma^0\gamma^{\mu_1}\gamma^0 \\ &= \gamma^0\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\gamma^0 \\ &= \gamma^0\Gamma'\gamma^0,\end{aligned}$$

where the third equality follows since all "interior" $\gamma^0$ matrices are in pairs, all of which multiply to 1. Using this, our desired result follows almost immediately:

$$\begin{aligned}(\overline{v}^s(p)\Gamma u^r(k))^\dagger &= \left(v^{s\dagger}(p)\gamma^0\Gamma u^r(k)\right)^\dagger \\ &= u^{r\dagger}(k)\Gamma^\dagger\gamma^0 v^s(p) \\ &= \overline{u}^r(k)\Gamma' v^s(p).\end{aligned}$$

**b)** Suppose that the scattering of a fermion off of some potential is described by the matrix element

$$i\mathcal{M} = iB_\mu\overline{u}^r(p')\gamma^\mu u^s(p),$$

where $p$ and $p'$ describe the initial and final momenta, respectively, and $s$ and $r$ the initial and final spins. $B_\mu$ contains the other numerical factors, that are unimportant for our purposes. Use the spin sum relation in Eq. (3.66) in P&S to show that the unpolarized, spin-averaged squared matrix element (this means that we average over the 2 possible initial-state spins and sum over the final-state ones) is given by

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{2} \sum_{s=1}^{2} \sum_{r=1}^{2} \left[ B_\mu \overline{u}^r(p') \gamma^\mu u^s(p) \right]^\dagger \left[ B_\nu \overline{u}^r(p') \gamma^\nu u^s(p) \right] \tag{12}$$

$$= \frac{1}{2} B_\mu^* B_\nu \mathrm{Tr}\left[ (\not{p}' + m) \gamma^\mu (\not{p} + m) \gamma^\nu \right], \tag{13}$$

where $m$ is the mass of the fermion and $\not{p} = p_\mu \gamma^\mu$.

**Hint:** One way of doing this is explained on p. 132 in P&S, but it is easier to use a) that a product of matrices that gives a scalar, or a $1 \times 1$ matrix can be written as its own trace, and b) the cyclic and linear properties of traces.

**Solution:** Using the result of part a, we first have

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{2} B_\mu^* B_\nu \sum_{s=1}^{2} \sum_{r=1}^{2} \overline{u}^s(p) \gamma^\mu u^r(p') \overline{u}^r(p') \gamma^\nu u^s(p).$$

Next we note that $\overline{u}^s(p)\gamma^\mu u^r(p')\overline{u}^r(p')\gamma^\nu u^s(p) = \mathrm{Tr}[\text{same expression}]$, and use the cyclic property of traces to move $u^s(p)$ from the back to the front; moving the sums into the trace (which we can do because of the linearity of traces) and using the aforementioned spin sum relation, we find

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{2} B_\mu^* B_\nu \mathrm{Tr}\left[ \left( \sum_{s=1}^{2} u^s(p) \overline{u}^s(p) \right) \gamma^\mu \left( \sum_{r=1}^{2} u^r(p') \overline{u}^r(p') \right) \gamma^\nu \right]$$

$$= \frac{1}{2} B_\mu^* B_\nu \mathrm{Tr}\left[ (\not{p}' + m) \gamma^\mu (\not{p} + m) \gamma^\nu \right].$$