University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 11

Problem 1 Scattering from a classical electromagnetic field (from the 2018 final exam) (T. Bringmann)

Let us describe the scattering of an electron, or positron, in a **time-independent classical electromagnetic field.** In Feynman diagrams (as you showed in the previous problem set), this is done by simply replacing the QED vertex rule $-ie\gamma^\mu \to -ie\gamma^\mu \tilde{A}_\mu$, where $\tilde{A}_\mu(\mathbf{q})$ is the Fourier transform of the classical electromagnetic potential; $q \equiv p_f - p_i$ is the difference between incoming and outgoing fermion momenta.

a) For an external potential that is not only time-independent but also localized in space, the scattering cross section can be written as

$$d\sigma = \frac{1}{2|\mathbf{p}_i|}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i \to p_f)|^2(2\pi)\delta(E_f - E_i). \tag{1}$$

Argue why this expression makes sense, and show that it is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}(p_i \to p_f)|^2. \tag{2}$$

b) Compute the scattering amplitude $\mathcal{M}$ for the scattering of an electron in the Coulomb potential created by a nucleus of charge Z, i.e. $A = (Ze/4\pi r, \mathbf{0})$. How does this expression look like for the scattering of a positron?

> **Hint:** The Fourier transform of the Coulomb potential is most easily calculated by adding a regulating factor $e^{-\mu r}$ to the potential, and then sending the 'photon mass' μ to zero at the end of the calculation.

c) Using the above expressions, calculate the spin-averaged differential cross-section for the scattering of an electron in a Coulomb potential, as a function of the scattering angle θ. The result is known as the *Mott formula.* Take the non-relativistic limit of this expression to obtain a well-known expression for the scattering of charged particles obtained earlier by *Rutherford.*

> **Hint:** You can simplify the resulting expression by using the trigonometric identity $1 - \cos\theta = 2\sin^2(\theta/2)$.

Problem 2 Kinematics (T. Bringmann – solutions: J. Van den Abeele)

This problem considers, in some detail, the phenomenologically very important relativistic kinematics of two-body reactions where two particles of four-momenta p_1 and p_2 and masses m_1 and m_2 in the initial state scatter to particles of momenta p_3 and p_4 and masses m_3 and m_4 in the final state.

a) How many *independent* Lorentz-invariant kinematic invariants can one form out of the four momenta p_i (i.e. how many Lorentz scalars that are not simply the masses of the involved particles)? In other words: how many kinematical degrees of freedom are required to describe the momenta of all involved particles?

What would be the answer for a $2 \to 3$ process?

b) The Lorentz-invariant *Mandelstam* variables are defined by

$$s \equiv (p_1 + p_2)^2 , \qquad t \equiv (p_1 - p_3)^2 , \qquad u \equiv (p_1 - p_4)^2 . \tag{3}$$

Show that they are not independent – as expected from a) – but satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 . \tag{4}$$

Problem 3 Bhabha scattering in the high-energy limit (T. Bringmann, T. Klungland – solutions: J. Van den Abeele, T. Klungland)

This problem considers the QED process of electron-positron scattering, $e^+e^- \to e^+e^-$, also known as *Bhabha scattering*.

a) Draw the two Feynman diagrams that describe this process to lowest order – i.e. $\mathcal{O}(\alpha^2)$, with $\alpha = e^2/4\pi$, in the cross section – and write down the corresponding amplitudes $\mathcal{M}_i$ for each diagram ($i = 1, 2$) in momentum representation.

b) When 'adding' these two diagrams to the total amplitude, $\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$, there is a relative minus sign. Derive this sign by working out the required contractions, and counting the number of fermion field commutations it takes to get the ordering of the contractions on the same form in both terms. As a rule of thumb it can also be seen directly from the Feynman diagrams, if you count the number of external fermion lines that need to be swapped for the two diagrams to be equivalent.

c) For high-energy scattering processes, i.e. where $s \gg m_e^2$, the electron masses can to a very good approximation be set to zero. Using this, and starting from Eq. (4.85) in P&S, show that the unpolarized cross-section (meaning that we take the average of the incoming spin states, and include all possible outgoing spin states by summing over these) for this process can be written differential in t as

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\langle|\mathcal{M}|^2\rangle. \tag{5}$$

Here $\langle|\mathcal{M}|^2\rangle$ is the squared total matrix element, averaged over initial-state spins and summing over final-state ones.

d) Calculate the leading-order differential cross-section for Bhabha scattering, using Eq. (5), in the ultrarelativistic limit $s \gg m_e^2$. Express the result in terms of the Mandelstam variables s, t, u.

You will notice the slightly alarming fact that this expression is divergent for $t = 0$, i.e. for $p_3 = p_1$ ($t = 0$ only implies $p_1 = p_3$ in the massless case like here, not in general); this divergence originates from the so-called t-channel diagram in the limit where the photon energy becomes extremely low. This is an example of a "soft" divergence, which is very common in this type of calculation involving massless particles (in this case it is the photon that causes problems); whenever a massless particle is radiated off or exchanged, the cross-section will be divergent in the limit where its energy goes to zero.

Luckily this isn't terribly relevant for practical cases; $t = 0$ implies $\cos\theta = 1$, i.e. forward scattering where the electron carries on in exactly the same direction. At a particle collider this means that the electron will continue down the tube where the positron came from without being detected; thus we will not actually observe any events with $t = 0$. In any case, the divergence will be canceled by including higher order diagrams.

Hint: The squared matrix element gets three terms, $\langle|\mathcal{M}|^2\rangle = \langle|\mathcal{M}_1|^2\rangle + \langle|\mathcal{M}_2|^2\rangle - 2\mathrm{Re}\langle\mathcal{M}_1\mathcal{M}_2^*\rangle$; evaluate each of these separately. The second term requires just a minimal amount of calculations if you recognize the similarities between the two contributing diagrams and make some appropriate substitutions.

The spin sums in each term can be manipulated similarly to the process on page 132 in P&S. However, notice that their manipulation is somewhat unnecessarily cumbersome; it is easier, in particular for the last term in the squared matrix element, to recognize that any number can be viewed as the trace of a 1×1 matrix, and then reorganize factors using the cyclic property of traces. For example, the expression in the first parenthesis in Eq. (5.2) of P&S can be manipulated as

$$\begin{aligned}\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p') &= \mathrm{Tr}\left[\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p')\right] \\ &= \mathrm{Tr}\left[v(p')\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu\right],\end{aligned}$$

which allows one to use completeness relations for the spinors when summing over spin states. It can also help to recognize that all factors of the form $\overline{v}\gamma^\mu u$, $\overline{u}\gamma^\nu u$, etc. commute since they are just numbers.

Finally, you will need contraction and trace identities for γ matrices derived in problem set 5.