University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 11

Problem 1 Scattering from a classical electromagnetic field (from the 2018 final exam)

(T. Bringmann)

Let us describe the scattering of an electron, or positron, in a **time-independent classical electromagnetic field.** In Feynman diagrams (as you showed in the previous problem set), this is done by simply replacing the QED vertex rule $-ie\gamma^\mu \to -ie\gamma^\mu \tilde{A}_\mu$, where $\tilde{A}_\mu(\mathbf{q})$ is the Fourier transform of the classical electromagnetic potential; $q \equiv p_f - p_i$ is the difference between incoming and outgoing fermion momenta.

a) For an external potential that is not only time-independent but also localized in space, the scattering cross section can be written as

$$d\sigma = \frac{1}{2|\mathbf{p}_i|}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i \to p_f)|^2(2\pi)\delta(E_f - E_i). \tag{1}$$

Argue why this expression makes sense, and show that it is equivalent to

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}(p_i \to p_f)|^2. \tag{2}$$

Solution: Compared to the standard phase-space, the conservation of 4-momentum is replaced by only energy conservation because in a time-independent setting energy must still be conserved (from Noether's theorem) — while the charged particles can change (the direction of) their momentum in the external field (because it is not translationally invariant). Compared to the common expression for $2 \to n$ scattering, the only difference is then a missing overall factor of $(2E_\mathcal{B})^{-1}$; this is the same difference as between the expressions for the cross section ($2 \to n$) and the decay rate ($1 \to n$).

$$\begin{aligned} d\sigma &= \frac{1}{2|\mathbf{p}_i|}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i \to p_f)|^2(2\pi)\delta(E_f - E_i) \\ &= \frac{1}{2|\mathbf{p}_i|}\frac{|\mathbf{p}_f|^2 d|\mathbf{p}_f| d\Omega}{(2\pi)^2}\frac{1}{2E_f}|\mathcal{M}(p_i \to p_f)|^2\delta(E_f - E_i) \\ &= \frac{|\mathbf{p}_f|}{4|\mathbf{p}_i|}\frac{dE_f d\Omega}{(2\pi)^2}|\mathcal{M}(p_i \to p_f)|^2\delta(E_f - E_i), \end{aligned}$$

where we have used $E^2 = |\mathbf{p}|^2 + m^2$ (i.e. $EdE = |\mathbf{p}|d|\mathbf{p}|$) in the last step. The δ function also enforces $|\mathbf{p}_f| = |\mathbf{p}_i|$, so after integrating over dE_f, we arrive at the desired result.

b) Compute the scattering amplitude $\mathcal{M}$ for the scattering of an electron in the Coulomb potential created by a nucleus of charge Z, i.e. $A = (Ze/4\pi r, \mathbf{0})$. How does this expression look like for the scattering of a positron?

Hint: The Fourier transform of the Coulomb potential is most easily calculated by adding a regulating factor $e^{-\mu r}$ to the potential, and then sending the 'photon mass' μ to zero at the end of the calculation.

Solution: The Coulomb potential of the (always positively charged) nucleus is given by

$$A^\mu(\mathbf{x}) = (Ze/4\pi r, 0, 0, 0),$$

so its Fourier transform is

$$\begin{aligned}\tilde{A}^0(\mathbf{q}) &= \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} A^0(\mathbf{x}) \\ &= \int_0^\infty r^2 dr \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\phi e^{-i|\mathbf{q}|r\cos\theta} \frac{Ze}{4\pi r} \\ &= \frac{Ze}{-2i|\mathbf{q}|} \int_0^\infty dr \Big(e^{-i|\mathbf{q}|r} - e^{i|\mathbf{q}|r}\Big).\end{aligned}$$

The integration over r has to be regulated (as we would get $e^{i\infty}$ and $e^{-i\infty}$ after integration). Let us introduce a regulating term $e^{-\mu r}$ (where $\mu > 0$) so that $e^{-\mu r} \to 0$ as $r \to \infty$. (*This prescription is actually physically well motivated because it corresponds to replacing the Coulomb potential with a Yukawa potential, i.e. the potential generated by a force carrier with mass μ.*)

$$\begin{aligned}A^0(\mathbf{q}) &= \lim_{\mu\to 0} \frac{Ze}{2i|\mathbf{q}|} \int_0^\infty dr \Big(e^{i|\mathbf{q}|r} - e^{-i|\mathbf{q}|r}\Big) e^{-\mu r} \\ &= \lim_{\mu\to 0} \frac{-Ze}{2i|\mathbf{q}|^2} \left(\frac{1}{i|\mathbf{q}| - \mu} + \frac{1}{i|\mathbf{q}| + \mu}\right) = \frac{Ze}{|\mathbf{q}|^2}.\end{aligned}$$

The amplitude for the scattering of electrons from this potential is thus given by

$$\begin{aligned}i\mathcal{M} &= \overline{u}^r(p_f)\Big(-ie\gamma^0 \tilde{A}_0(p_f - p_i)\Big) u^s(p_i) \\ &= -i\frac{Ze^2}{|\mathbf{q}|^2} \overline{u}^r(p_f)\gamma^0 u^s(p_i) \\ &= -i\frac{Ze^2}{|\mathbf{q}|^2} u^{r\dagger}(p_f) u^s(p_i).\end{aligned}$$

For positrons,

$$\begin{aligned}i\mathcal{M} &= \overline{v}^r(p_f)\Big(-ie\gamma^0 \tilde{A}_0(p_f - p_i)\Big) v^s(p_i) \\ &= -i\frac{Ze^2}{|\mathbf{q}|^2} v^{r\dagger}(p_i) v^s(p_f).\end{aligned}$$

By looking at the explicit expressions of u and v that we derived in the lecture, it is fairly straight-forward to see that the fermion bilinears are identical in the two cases, leading thus apparently to the same value of $\mathcal{M}$. However, the amplitude derives from contracting an expression of the form $A^\mu\langle f|\overline{\psi}\gamma^\mu\psi|i\rangle$. If the initial state is an electron (i.e. a fermion), both contractions can be performed without changing the location of the fermion operators. For an initial state positron, however, $\overline{\psi}$ has first to be moved past ψ before it can be contracted with $|i\rangle$. This gives a relative minus sign – which is exactly the expected difference between an attractive and repulsive Coulomb potential.

c) Using the above expressions, calculate the spin-averaged differential cross-section for the scattering of an electron in a Coulomb potential, as a function of the scattering angle θ. The result is known as the *Mott formula*. Take the non-relativistic limit of this expression to obtain a well-known expression for the scattering of charged particles obtained earlier by *Rutherford*.

Hint: You can simplify the resulting expression by using the trigonometric identity $1 - \cos\theta = 2\sin^2(\theta/2)$.

Solution: We have to sum over all possible spin configurations of the initial and final states, and then divide by a factor of $(2S_i + 1) = 2$ to account for the *averaging* over initial state configurations:

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{16\pi^2}\frac{1}{2}\sum_{r,s}|\mathcal{M}|^2 = \frac{1}{32\pi^2}\left(\frac{Ze^2}{|\mathbf{q}|^2}\right)^2\sum_{r,s}\left|\overline{u}^r(p_f)\gamma^0u^s(p_i)\right|^2 \\
&= \frac{\alpha^2Z^2}{2|\mathbf{q}|^4}\mathrm{tr}\left[(\not{p}_f + m)\gamma^0(\not{p}_i + m)\gamma^0\right] \\
&= \frac{\alpha^2Z^2}{2|\mathbf{q}|^4}\left(\mathrm{tr}\left[\not{p}_f\gamma^0\not{p}_i\gamma^0\right] + m^2\mathrm{tr}\left[\gamma^0\gamma^0\right]\right) \\
&= \frac{2\alpha^2Z^2}{|\mathbf{q}|^4}\left[p_{i\mu}p_{f\nu}\left(g^{\mu0}g^{0\nu} - g^{\mu\nu}g^{00} + g^{\mu0}g^{0\nu}\right) + m^2\right] \\
&= \frac{2\alpha^2Z^2}{|\mathbf{q}|^4}\left[2E_iE_f - p_i\cdot p_f + m^2\right] = \frac{2\alpha^2Z^2}{|\mathbf{q}|^4}\left[E_iE_f + \mathbf{p}_i\cdot\mathbf{p}_f + m^2\right].
\end{aligned}
$$

At this point, we recall that $E_f = E_i \equiv E$, and thus $|p_i| = |p_f| \equiv |p| = vE$ (where v is the velocity of the scattering electron). So,

$$
\begin{aligned}
\mathbf{p}_i\cdot\mathbf{p}_f &= v^2E^2\cos\theta = v^2E^2\left(1 - 2\sin^2\frac{\theta}{2}\right), \\
|\mathbf{q}|^2 = |\mathbf{p}_f - \mathbf{p}_i|^2 &= 2v^2E^2(1 - \cos\theta) = 4v^2E^2\sin^2\frac{\theta}{2}.
\end{aligned}
$$

Plugging this into the above expression, and using $m^2 = E^2(1 - v^2)$, gives

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= 2\alpha^2 Z^2 \left(\frac{1}{4v^2E^2\sin^2(\theta/2)}\right)^2 \left[E^2 + v^2E^2\left(1 - 2\sin^2\frac{\theta}{2}\right) + m^2\right] \\ &= 4E^2\alpha^2 Z^2 \left(\frac{1}{4v^2E^2\sin^2(\theta/2)}\right)^2 \left[1 - v^2\sin^2\frac{\theta}{2}\right] \\ &= \frac{\alpha^2 Z^2}{4v^4E^2}\frac{1 - v^2\sin^2(\theta/2)}{\sin^4(\theta/2)},\end{aligned}$$

which is the **Mott** formula. In the non-relativistic limit; $v \ll 1$ and hence $E \simeq m$, we therefore recover the **Rutherford** formula:

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4m^2v^4\sin^4(\theta/2)}.$$

Problem 2 Kinematics (T. Bringmann – solutions: J. Van den Abeele)

This problem considers, in some detail, the phenomenologically very important relativistic kinematics of two-body reactions where two particles of four-momenta p_1 and p_2 and masses m_1 and m_2 in the initial state scatter to particles of momenta p_3 and p_4 and masses m_3 and m_4 in the final state.

a) How many *independent* Lorentz-invariant kinematic invariants can one form out of the four momenta p_i (i.e. how many Lorentz scalars that are not simply the masses of the involved particles)? In other words: how many kinematical degrees of freedom are required to describe the momenta of all involved particles?

What would be the answer for a $2 \to 3$ process?

Solution: Let's consider a $2 \to N$ process. The number of degrees of freedom is given by the number of 4-momentum components p^μ of the involved particles, $4(N+2)$, minus the number of constraints:

1. $N+2$ on-shell conditions: $p_i^2 = m_i^2$
2. 4 4-momentum conservation conditions, $p_1^\mu + p_2^\mu = \sum_{i=1}^N k_i^\mu$
3. 6 conditions from the Lorentz invariance of the system (3 independent rotations and 3 boosts). For example, rotating a pair of four-vectors in spacetime by a common angle would not change their dot product.

Thus, the system has $4(N+2) - (N+2) - 4 - 6 = 3N - 4$ independent degrees of freedom.

For a $2 \to 2$ process, we have 2 degrees of freedom. One of the options for these 2 independent quantities is *any pair* of Mandelstam variables. For a $2 \to 3$ process we have 5 degrees of freedom.

b) The Lorentz-invariant *Mandelstam* variables are defined by

$$s \equiv (p_1 + p_2)^2\,, \qquad t \equiv (p_1 - p_3)^2\,, \qquad u \equiv (p_1 - p_4)^2\,. \tag{3}$$

Show that they are not independent – as expected from a) – but satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2\,. \tag{4}$$

Solution: From the definition of Mandelstam variables we have $s \equiv (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2$, $t \equiv (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$ and $u \equiv (p_1 - p_4)^2 = m_1^2 + m_4^2 - 2p_1 \cdot p_4$. From this we can easily calculate,

$$\begin{aligned} s + t + u &= m_1^2 + m_2^2 + 2p_1 \cdot p_2 + m_1^2 + m_3^2 - 2p_1 \cdot p_3 \\ &\qquad\qquad + m_1^2 + m_4^2 - 2p_1 \cdot p_4 \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 \cdot (p_2 - p_3 - p_4) \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 \cdot (-p_1) \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2. \end{aligned}$$

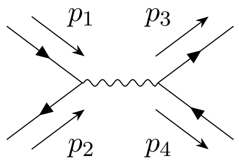
Problem 3 Bhabha scattering in the high-energy limit (T. Bringmann, T. Klungland – solutions: J. Van den Abeele, T. Klungland)

This problem considers the QED process of electron-positron scattering, $e^+e^- \to e^+e^-$, also known as *Bhabha scattering*.

a) Draw the two Feynman diagrams that describe this process to lowest order – i.e. $\mathcal{O}(\alpha^2)$, with $\alpha = e^2/4\pi$, in the cross section – and write down the corresponding amplitudes $\mathcal{M}_i$ for each diagram ($i = 1, 2$) in momentum representation.

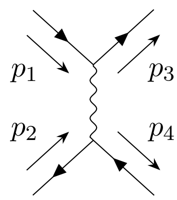
Solution: To lowest order, $\mathcal{O}(\alpha)$, there are two contributing QED diagrams. Using $s = (p_1 + p_2)^2$ and $t = (p_3 - p_1)^2$, their amplitudes are given by:

$i\mathcal{M}_s =$



$$= \overline{v}(p_2)(-ie\gamma^\mu)u(p_1)\left(\frac{-ig_{\mu\nu}}{s + i\epsilon}\right)\overline{u}(p_3)(-ie\gamma^\nu)v(p_4)$$

$i\mathcal{M}_t =$



$$= \overline{u}(p_3)(-ie\gamma^\mu)u(p_1)\left(\frac{-ig_{\mu\nu}}{t + i\epsilon}\right)\overline{v}(p_2)(-ie\gamma^\nu)v(p_4)$$

Note that in the full Standard Model, there would be similar diagrams with Z and h bosons mediating the process instead of a photon. Furthermore, there is no u-channel diagram, as the final-state particles e^- and e^+ are not identical.

b) When 'adding' these two diagrams to the total amplitude, $\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$, there is a relative minus sign. Derive this sign by working out the required contractions, and counting the number of fermion field commutations it takes to get the ordering of the contractions on the same form in both terms. As a rule of thumb it can also be seen directly from the Feynman diagrams, if you count the number of external fermion lines that need to be swapped for the two diagrams to be equivalent.

Solution: The s-channel diagram comes from the set of contractions

$$\mathcal{M}_s \sim \langle p_3, p_4|\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\overline{\psi}(y)\gamma^\nu\psi(y)A_\nu(y)|p_1, p_2\rangle,$$

while the t-channel one comes from

$$\mathcal{M}_t \sim \langle p_3, p_4|\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\overline{\psi}(y)\gamma^\nu\psi(y)A_\nu(y)|p_1, p_2\rangle.$$

In both cases, x and y are dummy variables that are integrated over. To see the relative minus sign between these two expressions, notice that if we move the $\psi(y)$ field (in the $\mathcal{M}_t$ expression) past $\overline{\psi}(y)$ and $\psi(x)$, and then swap the order of $\psi(x)$ and $\overline{\psi}(y)$, the structure of the contractions is the same in both expressions. This took three permutations of fermion fields; thus there is a relative factor $(-1)^3 = -1$.

From the Feynman diagrams, one can transform the s-channel diagram into the t-channel by interchanging the incoming e^- line and the outgoing e^+ line. This involves one permutation of fermion lines, indicating a relative factor of -1.

c) For high-energy scattering processes, i.e. where $s \gg m_e^2$, the electron masses can to a very good approximation be set to zero. Using this, and starting from Eq. (4.85) in P&S, show that the unpolarized cross-section (meaning that we take the average of the incoming spin states, and include all possible outgoing spin states by summing over these) for this process can be written differential in t as

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\langle|\mathcal{M}|^2\rangle. \tag{5}$$

Here $\langle|\mathcal{M}|^2\rangle$ is the squared total matrix element, averaged over initial-state spins and summing over final-state ones.

Solution: From P&S Eq. (4.85) we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 E_{\text{cm}}^2}.$$

Recognizing that $s = E_{\text{cm}}^2$ and that the process is symmetric around the collision axis, we can integrate over the solid angles $d\Omega = d\phi d(\cos\theta)$ as

$$\sigma = \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} \int_{-1}^{1} d(\cos\theta) \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s}.$$

Now, using that $t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 = \frac{s}{2}(\cos\theta - 1)$, we can change integration variables to obtain

$$\sigma = \int dt \frac{\langle|\mathcal{M}|^2\rangle}{16\pi s^2}.$$

Noting that $\sigma = \int d\sigma$, we then get the desired expression:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\langle|\mathcal{M}|^2\rangle.$$

d) Calculate the leading-order differential cross-section for Bhabha scattering, using Eq. (5), in the ultrarelativistic limit $s \gg m_e^2$. Express the result in terms of the Mandelstam variables s, t, u.

You will notice the slightly alarming fact that this expression is divergent for $t = 0$, i.e. for $p_3 = p_1$ ($t = 0$ only implies $p_1 = p_3$ in the massless case like here, not in general); this divergence originates from the so-called t-channel diagram in the limit where the photon energy becomes extremely low. This is an example of a "soft" divergence, which is very common in this type of calculation involving massless particles (in this case it is the photon that causes problems); whenever a massless particle is radiated off or exchanged, the cross-section will be divergent in the limit where its energy goes to zero.

Luckily this isn't terribly relevant for practical cases; $t = 0$ implies $\cos\theta = 1$, i.e. forward scattering where the electron carries on in exactly the same direction. At a particle collider this means that the electron will continue down the tube where the positron came from without being detected; thus we will not actually observe any events with $t = 0$. In any case, the divergence will be canceled by including higher order diagrams.

Hint: The squared matrix element gets three terms, $\langle|\mathcal{M}|^2\rangle = \langle|\mathcal{M}_1|^2\rangle + \langle|\mathcal{M}_2|^2\rangle - 2\text{Re}\langle\mathcal{M}_1\mathcal{M}_2^*\rangle$; evaluate each of these separately. The second term requires just a minimal amount of calculations if you recognize the similarities between the two contributing diagrams and make some appropriate substitutions.

The spin sums in each term can be manipulated similarly to the process on page 132 in P&S. However, notice that their manipulation is somewhat unnecessarily cumbersome; it is easier, in particular for the last term in the squared matrix element, to recognize that any number can be viewed as the trace of a 1×1 matrix, and then reorganize factors using the cyclic property of traces. For example, the expression in the first parenthesis in Eq. (5.2) of P&S can be manipulated as

$$\begin{aligned}\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p') &= \mathrm{Tr}\big[\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p')\big]\\ &= \mathrm{Tr}\big[v(p')\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu\big],\end{aligned}$$

which allows one to use completeness relations for the spinors when summing over spin states. It can also help to recognize that all factors of the form $\overline{v}\gamma^\mu u$, $\overline{u}\gamma^\nu u$, etc. commute since they are just numbers.

Finally, you will need contraction and trace identities for γ matrices derived in problem set 5.

Solution: We begin by evaluating the spin-averaged squared matrix element for the s-channel diagram, $\langle[\mathcal{M}_s|^2\rangle$. Labeling the spin states by $e^-(s)e^+(r) \to e^-(s')e^+(r')$, with each label either 1 or 2, averaging over the initial spins and summing over the final-state ones, we have

$$\begin{aligned}\langle|\mathcal{M}_s|^2\rangle &= \frac{1}{4}\sum_{r,s,r',s'=1}^{2}|\mathcal{M}|^2\\ &= \frac{e^4}{4s^2}\sum_{r,s,r',s'=1}^{2}\Big[\overline{v}^r(p_2)\gamma^\mu u^s(p_1)\overline{u}^s(p_1)\gamma^\nu v^r(p_2)\Big]\\ &\qquad\times\Big[\overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\nu u^{s'}(p_3)\Big]\\ &= \frac{e^4}{4s^2}\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu]\mathrm{Tr}[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu].\end{aligned}$$

The first trace evaluates to

$$\begin{aligned}\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu] &= p_{1\rho}p_{2\sigma}\mathrm{Tr}[\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu]\\ &= 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}p_1\cdot p_2).\end{aligned}$$

Evaluating the second trace similarly we find, after some algebra,

$$\begin{aligned}\langle|\mathcal{M}_s|^2\rangle &= \frac{8e^4}{s^2}[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4)]\\ &= 2e^4\left(\frac{u^2+t^2}{s^2}\right),\end{aligned}$$

where in the last line we have used that $t = -2p_1\cdot p_3 = -2p_2\cdot p_4$ and $u = -2p_1\cdot p_4 = -2p_2\cdot p_3$ for massless particles.

The square of the t-channel diagram can be found by inspection from the s-channel one; from part a we see that one can be converted into the other by interchanging p_2 and p_3 (if the electrons were not massless this would be slightly more complicated, as the completeness relation for the spinors has an additional mass term with sign depending on the particle/antiparticle nature of the spinor; but in our case this is not a problem); this reduces to $t \leftrightarrow s$, so that

$$\langle|\mathcal{M}_t|^2\rangle = 2e^4\left(\frac{u^2+s^2}{t^2}\right).$$

Finally there is the interference term, which is given by (as was noted in the hint, factors that are just numbers can be switched around freely)

$$\begin{aligned}-2\mathrm{Re}\langle \mathcal{M}_s \mathcal{M}_t^* \rangle &= -\frac{e^4}{2st} \sum_{r,s,r',s'=1}^{2} \overline{v}^r(p_2)\gamma^\mu u^s(p_1)\overline{u}^s(p_1)\gamma^\nu u^{s'}(p_3) \\ &\qquad \times \overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\overline{v}^{r'}(p_4)\gamma_\nu v^r(p_2) \\ &= -\frac{e^4}{2st}\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu].\end{aligned}$$

We evaluate this trace by first using contraction identities: Factoring out the momenta from the "slashed" matrices by $\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu = p_{2\alpha}p_{1\beta}p_{3\sigma}p_{4\rho}\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\sigma\gamma_\mu\gamma^\rho\gamma_\nu$ we have

$$\begin{aligned}\gamma^\alpha\left(\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\sigma\gamma_\mu\right)\gamma^\rho\gamma_\nu &= -2\gamma^\alpha\gamma^\sigma\left(\gamma^\nu\gamma^\beta\gamma^\rho\gamma_\nu\right) \\ &= -8g^{\beta\rho}\gamma^\alpha\gamma^\sigma.\end{aligned}$$

We then find

$$\begin{aligned}-2\mathrm{Re}\langle \mathcal{M}_s \mathcal{M}_t^* \rangle &= \frac{4e^4}{st}(p_1\cdot p_4)p_{2\alpha}p_{3\sigma}\mathrm{Tr}[\gamma^\alpha\gamma^\sigma] \\ &= \frac{16e^4}{st}(p_1\cdot p_4)(p_2\cdot p_3) \\ &= 4e^4\frac{u^2}{st}.\end{aligned}$$

Finally, combining the three terms we have found, reorganizing a bit, and inserting everything into Eq. (5), we get the unpolarized differential cross-section:

$$\frac{d\sigma}{dt} = \frac{e^4}{8\pi s^2}\left(\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2\right).$$