University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 12

Problem 1 Electron vertex function (T. Klungland)

This problem fills in some of gaps in the calculation in chapter 6.3 in P&S, specifically the numerator algebra on pages 191–192.

a) Derive the general version of Eqs. (6.45) and (6.46) for d-dimensional integrals: Specifically,

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu}{D^3} = 0, \tag{1}$$

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{D^3} = \int \frac{d^d\ell}{(2\pi)^d}\frac{\frac{1}{d}g^{\mu\nu}\ell^2}{D^3}, \tag{2}$$

where D only depends on ℓ^2 and a generic Lorentz scalar Δ.

Solution: To show Eq. (1), we transform $\ell \to -\ell$; this means that $d^d\ell \to (-1)^d d^d\ell$, and each of the d integrals get their limits changed to $(\infty, -\infty)$. Flipping these limits around we pick up another factor $(-1)^d$; thus we have

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu}{D^3} = -\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu}{D^3} = 0,$$

since D is invariant under this transformation (another, possibly simpler, argument for the two factors $(-1)^d$ canceling is to use that the Jacobian for the transformation is 1).

For Eq. (2), we recognize that the integral is a rank-2 Lorentz tensor; thus it must be proportional to $g^{\mu\nu}$ as this is the only rank-2 tensor available (since the integral does not depend on any other such tensors; after integrating out ℓ the integral only depends on the generic scalar Δ). We therefore have

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{D^3} = Cg^{\mu\nu},$$

where C is some undetermined Lorentz scalar. To solve for C we contract the indices on both sides, using $g^{\mu\nu}g_{\mu\nu} = \mathrm{Tr}(\mathbb{1}_{d\times d}) = d$, to find

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{D^3} = Cd.$$

Solving for C then gives Eq. (2).

---

b) Using the above results with $d = 4$, show that the numerator of the loop integral can be taken to

$$\begin{aligned}\text{Numerator} &= \overline{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p) \\ &\to \overline{u}(p')\Big[-\frac{1}{2}\gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\ &\qquad + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)\Big]u(p), \qquad (3)\end{aligned}$$

where $\ell = k + yq - zp$ and $k' = k + q$.

Solution: Substituting out k and k', we have

$$\begin{aligned}\text{Numerator} &= \overline{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p) \\ &= \overline{u}(p')[(\not{\ell} - y\not{q} + z\not{p})\gamma^\mu(\not{\ell} + (1-y)\not{q} + z\not{p}) \\ &\qquad + m^2\gamma^\mu - 2m(2\ell^\mu + (1-2y)q^\mu + 2zp^\mu)]u(p).\end{aligned}$$

All terms with ℓ vanish unless there are two factors, by the symmetry arguments in the previous part. The remaining ℓ dependent term can be rewritten by first using Eq. (2) and then a contraction identity:

$$\begin{aligned}\not{\ell}\gamma^\mu\not{\ell} &= g_{\rho\sigma}g_{\alpha\beta}\underbrace{\ell^\rho\ell^\alpha}_{\to\frac{1}{4}g^{\rho\alpha}\ell^2}\gamma^\sigma\gamma^\mu\gamma^\beta \to \frac{1}{4}\ell^2\underbrace{\gamma^\sigma\gamma^\mu\gamma_\sigma}_{=-2\gamma^\mu} \\ &= -\frac{1}{2}\ell^2\gamma^\mu.\end{aligned}$$

Inserting this and canceling all terms that are linear in ℓ, we recover the Eq. (3).

---

c) The numerator can be further simplified by using the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and the Dirac equation $\not{p}u(p) = mu(p)$, $\overline{u}(p')\not{p}' = \overline{u}(p')m$ ($\Rightarrow \overline{u}(p')\not{q}u(p) = 0$ since $q = p' - p$). As this is a profoundly boring calculation I won't ask you to go through all of it in detail to reproduce the expression at the top of page 192 in P&S;[1] instead, to get some practice with the type of Dirac matrix manipulations that are required in these calculations, show the following:

$$\overline{u}(p')[\not{q}\gamma^\mu\not{q}]u(p) = -\overline{u}(p')\left[q^2\gamma^\mu\right]u(p), \qquad (4)$$
$$\overline{u}(p')[\not{p}\gamma^\mu\not{p}]u(p) = \overline{u}(p')\left[2mp^\mu - m^2\gamma^\mu\right]u(p). \qquad (5)$$

[1] You *can*, of course, I just don't want to type out the full solution. Some tips to recover the expression in P&S if you do decide to go through the calculation: After the main Dirac algebra calculation, use $p = \frac{1}{2}((p+p') - q)$ and $p' = \frac{1}{2}((p+p') + q)$ to get the same three main terms as the book; the rest is just reorganizing using $x + y + z = 1$.

Solution: To show Eq. (4), we first use $\not{q}\gamma^\mu = 2q^\mu - \gamma^\mu\not{q}$; with $\overline{u}(p')\not{q}u(p) = 0$ and $(\not{q})^2 = q^2$, we then have

$$\begin{aligned}\overline{u}(p')\left[\not{q}\gamma^\mu\not{q}\right]u(p) &= \overline{u}(p')\left[2q^\mu\not{q} - \gamma^\mu\not{q}\not{q}\right]u(p)\\ &= -\overline{u}(p')\left[q^2\gamma^\mu\right]u(p).\end{aligned}$$

Eq. (5) is evaluated similarly, but here we have $\not{p}u(p) = mu(p)$ and $(\not{p}) = p^2 = m^2$, giving

$$\begin{aligned}\overline{u}(p')\left[\not{p}\gamma^\mu\not{p}\right]u(p) &= \overline{u}(p')\left[2p^\mu\not{p} - \gamma^\mu\not{p}\not{p}\right]u(p)\\ &= \overline{u}(p')\left[2mp^\mu - m^2\gamma^\mu\right]u(p).\end{aligned}$$

---

d) The final step to find the form factors F_1, F_2 (apart from the loop integral itself, of course) uses the Gordon identity:

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p), \tag{6}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Prove this identity.

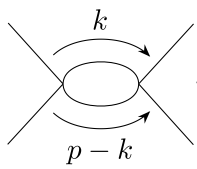
Solution: The simplest way to show Eq. (6) is to calculate the right-hand side; inserting the definitions of $\sigma^{\mu\nu}$ and q we have, using the Dirac algebra and -equation as in the previous part:

$$\begin{aligned}&\overline{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)\\ &= \overline{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} - \frac{1}{4m}\left(\gamma^\mu\left(\not{p}' - \underbrace{\not{p}}_{\to m}\right) - \left(\underbrace{\not{p}'}_{\to m} - \not{p}\right)\gamma^\mu\right)\right]u(p)\\ &= \overline{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{1}{2}\gamma^\mu - \frac{1}{4m}\left(2p'^\mu - \underbrace{\not{p}'}_{\to m}\gamma^\mu + 2p^\mu - \gamma^\mu\underbrace{\not{p}}_{\to m}\right)\right]u(p)\\ &= \overline{u}(p')\gamma^\mu u(p).\end{aligned}$$

Problem 2 1-loop renormalization of ϕ^4 theory (T. Klungland)

This problem is intended to give some practical context for some of the important ingredients in loop calculations — Feynman parameters, Wick rotations, regularization and renormalization — through an example. We will consider one of the 1-loop corrections to $2 \to 2$ scattering in ϕ^4 theory that were discussed in problem set 8; for simplicity we ignore the t- and u-channel diagrams, and assume the particles to be massless.

The leading-order diagram is given by $i\mathcal{M}_1 = -i\lambda_0$, where we have renamed the *bare* Lagrangian parameter to λ_0 for later convenience. At the next-to-leading order, we have

$i\mathcal{M}_2 =$  , (7)

with $p = p_1 + p_2$ so that $p^2 = s$(the incoming momenta are $p_{1,2}$; the outgoing ones are $p_{3,4}$).

a) Write down the amplitude for this diagram using the Feynman rules on p. 115 in P&S.

Solution:

$$i\mathcal{M}_2 = \frac{(-i\lambda_0)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{(k^2 + i\varepsilon)\Big((p-k)^2 + i\varepsilon\Big)}.$$

---

b) We evaluate this integral in two steps. Introduce a Feynman parameter x and use the identity

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}; \tag{8}$$

then complete the square in the denominator to rewrite the matrix element as

$$i\mathcal{M}_2 = \frac{\lambda_0^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\Big[(k-xp)^2 - \Delta + i\varepsilon\Big]^2}; \tag{9}$$

find an expression for Δ that is independent of k.

Note that we have actually cheated here; as you will see shortly the momentum integral is divergent, meaning that the interchange of integral orders, that we have done here, is illegitimate. We will fix this shortly by introducing a regulator of the momentum integral.

Solution: We can write the matrix element as

$$i\mathcal{M}_2 = \frac{\lambda_0^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{D^2},$$

where

$$\begin{aligned} D &= x\Big((p-k)^2 + i\varepsilon\Big) + (1-x)\big(k^2 + i\varepsilon\big) \\ &= k^2 - 2xk\cdot p + xp^2 + i\varepsilon \\ &= (k-xp)^2 - \Delta + i\varepsilon, \end{aligned}$$

with $\Delta = -sx(1-x)$.

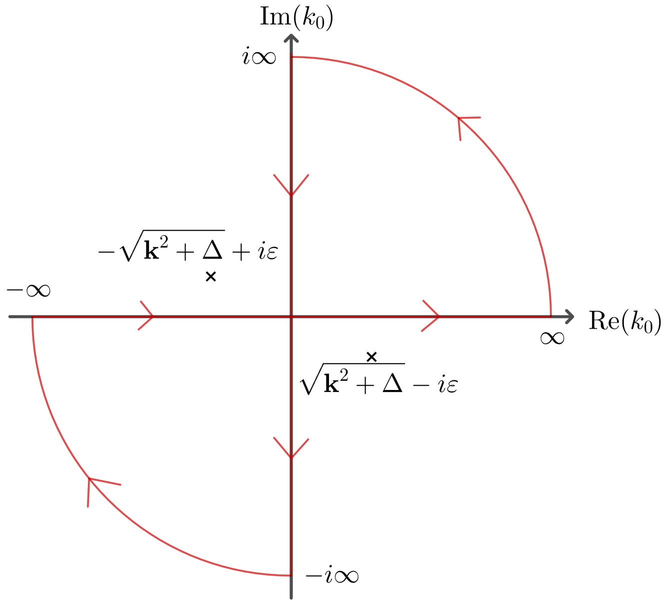


Figure 1: The integration contour C described in the problem text, shown in red.

We can now make a variable transformation in the integral, taking $k \to k + xp$, to get

$$i\mathcal{M}_2(p) = \frac{\lambda_0^2}{2}\int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^2}. \tag{10}$$

The integral is now spherically symmetric, but still on a form that is cumbersome to evaluate, given the Lorentz signature $k^2 = (k_0)^2 - \mathbf{k}^2$. We can fix this by using our second step, called Wick rotation, wherein we use the Feynman "$+i\varepsilon$" prescription of the propagators; in the complex k^0 plane, the poles of the integrand are located at

$$k_0 = \pm\sqrt{\mathbf{k}^2+\Delta} \mp i\varepsilon. \tag{11}$$

With this in mind, we define a closed integration contour for k^0 where we integrate along the real axis from $-\infty$ to ∞; in a quarter-circle from ∞ to $i\infty$; along the imaginary axis from $i\infty$ to $-i\infty$; and in another quarter-circle from $-i\infty$ to $-\infty$. A sketch of the contour and the poles is shown in Fig. 1.

c) Evaluate the k_0 integral:

$$\oint_C \frac{dk_0}{2\pi}\frac{1}{\left((k_0)^2-\mathbf{k}^2-\Delta+i\varepsilon\right)^2}, \tag{12}$$

where C is shown in Fig. 1 (this should not require any calculation). Use the result to argue that we can rotate the integration contour in Eq. (10) by 90 degrees counterclockwise; then make a variable change to the *Euclidean* momentum $k_E = \left(k_E^0, \mathbf{k}_E\right) \equiv (-ik_0, \mathbf{k})$, which satisfies $k_E^2 = \left(k_E^0\right)^2 + \mathbf{k}_E^2$.

Keep the $i\varepsilon$ regulator in the denominator; its main purpose has been served in allowing us to perform the Wick rotation, but it will still be needed to regulate the integral over the Feynman parameter x later.

Solution: Since the integrand in Eq. (12) has no poles within this contour, the integral is zero, which again means that the integrals along the real and imaginary axes exactly cancel since the quarter-circle parts, being infinitely far away, do not contribute.

In other words, we get exactly the same result by changing the integration limits on k_0 from $(-\infty,\infty)$ to $(-i\infty,i\infty)$. We can then make a variable change to the Euclidean momentum $k_E = \left(k_E^0, \mathbf{k}_E\right) \equiv (-ik_0, \mathbf{k})$, so that $d^4k = id^4k_E$ and $k^2 = -\left(k_E^0\right)^2 - \mathbf{k}_E^2 = -k_E^2$.

After changing variables our expression has the form of a spherically symmetric integral:

$$i\mathcal{M}_2 = \frac{i\lambda_0^2}{2}\int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{\left(k_E^2+\Delta-i\varepsilon\right)^2}.$$

We now arrive at a problem, since the momentum integral is divergent in the limit $k_E \to \infty$, i.e. the ultraviolet (UV) limit. This can be fixed in a number of ways; here we choose what is called *dimensional regularization*. What this means is that we write the integral as a function of the number of spacetime dimensions d, and write the divergences as poles in $\epsilon = \frac{1}{2}(4-d)$. In practice this boils down to changing the dimension of the above momentum integral from 4 to $d = 4-2\epsilon$; also, to keep the coupling dimensionless,[2] we make the transformation $\lambda_0 \to \mu^{4-d}\lambda_0$, where μ is some arbitrary energy scale. This gives the matrix element an overall factor of μ^{4-d}, as the leading-order matrix element now becomes $-i\lambda_0\mu^{4-d}$. Since this extra factor is ultimately uninteresting for our purposes, and will not contribute when we take the limit $d \to 4$, we can simply leave it out.[3] Our matrix element then reads

$$i\mathcal{M}_2 = \frac{i\lambda_0^2}{2}\mu^{4-d}\int_0^1 dx \int \frac{d^dk_E}{(2\pi)^d}\frac{1}{\left(k_E^2+\Delta-i\varepsilon\right)^2}. \tag{13}$$

Evaluating the momentum integral itself it somewhat tedious; it can be done using the Euler beta function, which is given by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx x^{a-1}(1-x)^{b-1}. \tag{14}$$

To begin we introduce spherical coordinates $d^dk_E = d\Omega_d k_E^{d-1}dk_E$, with

$$d\Omega_d = \sin^{d-2}\phi_{d-1}\sin^{d-3}\phi_{d-3}\cdots\sin\phi_2 d\phi_1\cdots d\phi_{d-1}. \tag{15}$$

The integration limits on the various angles are $\phi_1 \in [0,2\pi)$, $\phi_i \in [0,\pi)$ for $i > 1$; defining the variables $x_i = \sin^2\phi_i$, the d-dimensional surface integral

[2] You will see the need for this in QFT2.

[3] If you are unsatisfied by this, you can replace $i\mathcal{M} \to i\mathcal{M}/\mu^{4-d}$ in the following expressions.

evaluates to

$$\Omega_d = \int d\Omega_d = 2\pi \prod_{i=2}^{d-1} \left(\int_0^\pi d\phi_i \sin^{i-1}\phi_i \right) \tag{16}$$
$$= 2\pi \prod_{i=2}^{d-1} \left(\int_0^1 dx x^{\frac{i}{2}-1}(1-x)^{-\frac{1}{2}} \right)$$
$$= 2\pi \prod_{i=2}^{d-1} \left(\frac{\Gamma\left(\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)} \right)$$
$$= 2\pi^{d/2} \frac{\Gamma\left(\frac{2}{2}\right)\Gamma\left(\frac{3}{2}\right)\cdots\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{4}{2}\right)\cdots\Gamma\left(\frac{d}{2}\right)} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}, \tag{17}$$

using $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.

The remaining k_E integral is also most conveniently done by a change of variables, defining $z \equiv \tilde{\Delta}/\left(k_E^2 + \tilde{\Delta}\right)$ where $\tilde{\Delta} \equiv \Delta - i\varepsilon$, which gives another Beta integral. Explicitly, we find

$$\int dk_E \frac{k_E^{d-1}}{\left(k_E^2 + \tilde{\Delta}\right)^2} = \frac{1}{2}\tilde{\Delta}^{\frac{d-4}{2}} \int_0^1 dz (1-z)^{\frac{d-2}{2}} z^{\frac{2-d}{2}}$$
$$= \frac{1}{2}\tilde{\Delta}^{\frac{d-4}{2}} \Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{4-d}{2}\right). \tag{18}$$

Writing out $\tilde{\Delta}$ and collecting factors then leaves

$$i\mathcal{M}_2 = \frac{i\lambda_0^2}{2(4\pi)^{\frac{d}{2}}} \left(\frac{\mu^2}{s}\right)^{\frac{4-d}{2}} \Gamma\left(\frac{4-d}{2}\right) \int_0^1 dx [-x(1-x) - i\varepsilon]^{\frac{d-4}{2}}. \tag{19}$$

d) Eq. (19) contains a pole at $d = 4$. We can expand around this pole by setting $d = 4 - 2\epsilon$, and expanding:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon), \tag{20}$$

where $\gamma_E \approx 0.577$ is the Euler-Mascheroni constant and $\epsilon > 0$. Use this, and the Taylor expansion $a^\epsilon = 1 + \epsilon \ln a + \mathcal{O}\left(\epsilon^2\right)$, to expand Eq. (19) in ϵ, dropping all terms of order ϵ or higher (which vanish in the physical limit $\epsilon \to 0$). To get rid of some irrelevant numerical terms, define $1/\bar{\epsilon} \equiv 1/\epsilon - \gamma_E + \ln 4\pi$.

Solution: Inserting $d = 4 - 2\epsilon$ and dropping terms of order ϵ or higher, we have

$$i\mathcal{M}_2 = \frac{i\lambda_0^2}{32\pi^2}\left[\frac{1}{\bar{\epsilon}} + \ln\frac{\mu^2}{s} - \int_0^1 dx \ln\left[-x(1-x) - i\varepsilon\right]\right],$$

where $1/\bar{\epsilon} \equiv 1/\epsilon - \gamma_E + \ln 4\pi$.

e) Perform the remaining integral over the Feynman parameter x, using $\ln(-x - i\varepsilon) = -i\pi + \ln(x + i\varepsilon)$. Note that you can take

$$-x(1-x) - i\varepsilon = (-x - i\varepsilon)(1 - x + i\varepsilon) + \mathcal{O}(\varepsilon^2),$$

where the $\mathcal{O}(\varepsilon^2)$ terms can be neglected, to simplify the integral. Adding the leading order contribution, you should find that the scattering amplitude is given by

$$i\mathcal{M} = -i\lambda_0 + \frac{i\lambda_0^2}{32\pi^2}\left[\frac{1}{\epsilon} + 2 + \ln\frac{\mu^2}{s} + i\pi\right] + \mathcal{O}(\lambda_0^3), \tag{21}$$

after taking the limit $\varepsilon \to 0$ for the "Feynman prescription parameter" ε (not to be confused with the dimensional regularization parameter ϵ).

Solution: First, we have $\ln[-x(1-x) - i\varepsilon] \simeq \ln(-x - i\varepsilon) + \ln(1 - x + i\varepsilon)$. Then, changing variables $x \to 1 - x$ in the second term, we get

$$\begin{aligned}\int_0^1 dx \ln[-x(1-x) - i\varepsilon] &= -i\pi + 2\int_0^1 dx \ln(x + i\varepsilon) \\ &= -i\pi - 2 + \mathcal{O}(\varepsilon),\end{aligned}$$

after which we can safely take $\varepsilon \to 0$.

This concludes the loop calculation, but we still have the divergence in Eq. (21) to deal with. This is done by renormalization, or a redefinition of the Lagrangian parameter λ_0 in terms of physically measured quantities. In doing this we make use of the fact that the parameters of the Lagrangian are not directly measurable, instead they are determined indirectly through scattering experiments, etc. We can thus define the Lagrangian parameters in terms of *renormalized* quantities. Of course, in order for our theory to be predictive we need this re-definition to be general; in other words, once we have fixed the parameters of the Lagrangian based on one process, we can use these parameters to calculate other processes to the same order, and also obtain finite results. This property is called *renormalizability,* and will be covered in QFT2.

Suppose that we can measure this scattering amplitude, and that we define a renormalized coupling λ by its value at some center-of-mass energy s_0:[4]

$$\lambda \equiv -\mathcal{M}(s = s_0). \tag{22}$$

To calculate the amplitude at other energies we assume that the bare coupling λ_0 can be expressed as a power series in λ:

$$\lambda_0 = \lambda + a\lambda^2 + \mathcal{O}(\lambda^3). \tag{23}$$

[4]Given that this is a made-up toy model this can be tough to interpret; suppose for the sake of the example that this model actually described nature. λ would then be regarded as a fundamental physical constant, and its experimentally determined value would be defined by a measurement at some particular energy scale $\sqrt{s_0}$, much like how the electron charge e in QED is related to the exchange of a low-energy photon (corresponding to a long-range Coulomb interaction).

f) Insert Eq. (23) into Eq. (21) for $s = s_0$, and solve for a, to find an expression for λ_0 to order $\mathcal{O}(\lambda^2)$. Finally, insert this expression into Eq. (21) for a general energy s to obtain a finite (in the limit $d \to 4$) result for the scattering amplitude to order $\mathcal{O}(\lambda^2)$.

Solution: Inserting Eq. (23) into Eq. (21) for $s = s_0$ gives

$$-\lambda = -\left(\lambda + a\lambda^2\right) + \frac{\lambda^2}{32\pi^2}\left[\frac{1}{\epsilon} + 2 + \ln\frac{\mu^2}{s_0} + i\pi\right] + \mathcal{O}\left(\lambda^3\right),$$

and solving for a yields

$$\lambda_0 = \lambda + \frac{\lambda^2}{32\pi^2}\left[\frac{1}{\epsilon} + 2 + \ln\frac{\mu^2}{s_0} + i\pi\right] + \mathcal{O}\left(\lambda^3\right).$$

Finally, we insert this into Eq. (21) for a general s, to obtain the finite result

$$\mathcal{M}(s) = -\lambda - \frac{\lambda^2}{32\pi^2}\ln\frac{s}{s_0} + \mathcal{O}\left(\lambda^3\right).$$