University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 13

Problem 1 Quantization and energies (2019 final exam, slightly modified) (T. Bringmann)

Let us start by revisiting one of the important **conceptual aspects** of quantum field theory.

a) Demonstrate that a naive quantization of relativistic particles, along the lines of how Schrödinger quantised a non-relativistic particle (*i.e. taking* $p^\mu \to i\partial^\mu$ *and enforcing the relativistic energy-mass relation* $p^2 = m^2$), leads to the existence of negative energies! Why is this not acceptable for a physical theory?

Solution: The heuristic way of quantizing non-relativistic particles is to promote the Hamiltonian, $H = p^2/(2m)$, to an operator by replacing $p \to -i\nabla$. The Schrödinger equation is then an energy eigenvalue equation for the wavefunction ψ, $E\psi = H\psi = -\nabla^2/(2m)\psi$. The corresponding Lorentz-invariant energy-momentum relation is $p^\mu p_\mu = m^2$, and the 'correct' quantization prescription matching the classical expression is $p^\mu \to i\partial^\mu$. This gives the Klein-Gordon equation,

$$(\partial^2 + m^2)\psi = 0,$$

with plane wave ($e^{\pm ik\cdot x}$) solutions that satisfy $k^2 = m^2$. Because $i\partial^\mu e^{\pm ik\cdot x} = \mp k^\mu e^{\pm ik\cdot x}$, the energy is given by $p^0 = \mp k^0 = \mp\sqrt{\mathbf{k}^2 + m^2}$. From the classical equations of motion, such a theory is necessarily unstable: particles would continue to increase their momentum without any bounds (and thereby lower their energy).

b) When adopting the canonical quantization prescription that we learned about in this course, we still encountered the same relativistic wave equation as one would expect from the argument in a). What is the fundamental difference, both conceptually (in the procedure) and in terms of interpretation (of the resulting equation of motion)? Explain (in words) how to obtain the Hamiltonian H of a scalar quantum field, and state it in a form where it becomes manifest that its eigenvalues are bounded from below!

Solution: The fundamental difference is that the Klein-Gordon equation now describes the evolution of a field operator ϕ rather than a quantum-mechanical wavefunction that this operator acts on; what we interpreted as negative-energy solutions above are thus simply negative frequency solutions (not related to negative eigenvalues of the Hamiltonian). The Hamiltonian for the quantum field is obtained by replacing the fields in the classical expression, $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$, with field operators. By introducing annihilation and creation operators, the resulting expression can be diagonalized to

$$H = \int d^3x \mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}},$$

which is manifestly positive because the harmonic oscillator frequencies $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$ are positive, and the eigenvalues of $N_{\mathbf{k}} = a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ are non-negative integers that describe the number of field excitations with momentum $\mathbf{k}$.

c) Explain how the requirement of $H > 0$ enforces particles described by the Dirac equation to be fermions! *[Following the same procedure as in b), introducing commutation relations for the fermion fields, would lead to the expression*

$$H = \int d^3x \mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1,2} \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^{s\dagger} a_{\mathbf{k}}^s - b_{\mathbf{k}}^{s\dagger} b_{\mathbf{k}}^s \right).$$

Identify the problem with this expression and argue that it is fixed by replacing the commutation relations with anti-commutation ones. Then show that the resulting creation/annihilate operators create/annihilate fermions.]

Solution: Quantizing a field ψ that satisfies the Dirac equation, by following the same procedure as outlined above, leads to the Hamiltonian

$$H = \int d^3x \mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1,2} \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^{s\dagger} a_{\mathbf{k}}^s - b_{\mathbf{k}}^{s\dagger} b_{\mathbf{k}}^s \right),$$

where $a_{\mathbf{k}}^{s\dagger}$ ($a_{\mathbf{k}}^s$) are the creation (annihilation) operators for particles, associated to the positive-frequency solutions of the Klein-Gordon equation with spin s; $b_{\mathbf{k}}^{s\dagger}$ and $b_{\mathbf{k}}^s$ describe those for anti-particles, associated to the negative-frequency solutions. These operators would inherit commutation relations from the (equal-time) field quantization prescription $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$.

Recognizing number operators as in b), the eigenvalues of H are thus *not* bounded from below. This can be solved by imposing *anti-commutation relations* $\{\phi(\mathbf{x}), \pi(\mathbf{y})\} = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$ instead, which translates to $\left\{ b_{\mathbf{k}}^s, b_{\mathbf{k}'}^{r\dagger} \right\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\mathbf{k} - \mathbf{k}')$. Unlike the previous case, this is an algebra that does not distinguish between $b_{\mathbf{k}}^s$ and $b_{\mathbf{k}}^{s\dagger}$. We can thus exchange $b_{\mathbf{k}}^s \leftrightarrow b_{\mathbf{k}}^{s\dagger}$ in the above Hamiltonian which, after employing the anti-commutation relation and discarding an infinite constant, leads to the manifestly positive Hamiltonian

$$H = \int d^3x \mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1,2} \omega_{\mathbf{k}} \left(a_{\mathbf{k}}^{s\dagger} a_{\mathbf{k}}^s + b_{\mathbf{k}}^{s\dagger} b_{\mathbf{k}}^s \right).$$

Creation operators satisfying anti-commutation relations describe *fermions* because the excitations respect the Pauli exclusion principle (no more than one state with identical quantum numbers is allowed),

$$\left(b_{\mathbf{k}}^{s\dagger} \right)^2 |0\rangle = 0,$$

and the exchange of two particles results in a relative minus sign,

$$b_{\mathbf{k}}^{s\dagger} b_{\mathbf{k}'}^{r\dagger} |0\rangle = -b_{\mathbf{k}'}^{r\dagger} b_{\mathbf{k}}^{s\dagger} |0\rangle.$$

Problem 2

a) What is a scalar? A pseudo-scalar? A vector? Pseudo-vector? Give examples.

Solution: See P&S Sec. 3.4 and 3.6.

b) Is the Lagrangian scalar? The Hamiltonian?

Solution: The Lagrangian *is* a scalar; the Hamiltonian is not (it is the 0'th component of the 4-momentum, thus it is a component of a vector).

c) Is the three-dimensional delta function a scalar? If yes, why? If no, how can you scale it such that it becomes scalar? Is $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}}$ scalar? Why/why not?

Solution: See P&S p. 23.

d) What are the conserved quantities associated with translational invariance (in space and time)?

Solution: Time $\rightarrow$ energy, space $\rightarrow$ spatial momentum.

e) What is the advantage of using the interaction picture in perturbation theory?

Solution: Fields propagate like free fields, while the time-dependence from the interaction part of the Hamiltonian only affects states. Thus correlation functions in an interacting theory can be expressed by the free propagators plus perturbations.

f) What does Wick's theorem tell you about the vacuum expectation value of products of fields?

Solution: See P&S Sec. 4.3.

g) Why do not the disconnected diagrams contribute to the correlation function?

Solution: See P&S Eq. (4.31); the sum of disconnected diagrams factor in the numerator, and the denominator gives the same sum of disconnected diagrams. Thus they cancel. (see p. 96–98)

h) The S matrix is defined as the overlap between in- and out-states. Are the in-state and/or out-state eigenstates of the Hamiltonian H? Of the free-field Hamiltonian H^0?

Solution: Single-particle momentum states are eigenstates of the full Hamiltonian H (not, in general, of the free one H^0). Multi-particle combinations of such states are *not* eigenstates of H, since the higher-order terms essentially rotate the single-momentum eigenstates into one another. They will not in general be eigenstates of H^0 either.

i) How can you obtain the S matrix elements from correlation functions?

Solution: LSZ formula, see P&S Sec. 7.2.

j) What are the similarities and differences between Yukawa theory and QED?

Solution: Yukawa theory describes a fermion/antifermion pair interacting with a scalar; in QED the scalar is replaced by a vector (photon). Phenomenologically, as shown in the table on p. 126 in P&S, Yukawa theory describes an attractive potential between any pair of fermions and/or antifermions; in QED the potential between equal charges is repulsive.

k) Why can we not quantize the photon field in the same way as fermion and scalar fields, and how is this dealt with?

Solution: See the midterm (problem set 6); the problem is that the photon field has an additional unphysical degree of freedom due to gauge invariance. It is typically fixed by choosing a particular gauge.

l) Can you build a laser producing longitudinal or scalar photons?

Solution: No; as you showed in the midterm only the transverse photon polarizations contribute to physical observables.

m) What is Ward's identity? Mention one application of it.

Solution: The Ward identity is the (Feynman) diagrammatic representation of gauge invariance. An amplitude containing a photon with momentum q^μ and polarization $\epsilon^\mu(q)$ can be written as $\mathcal{M} = \epsilon_\mu(q)\mathcal{M}^\mu$; the identity then states that $q_\mu \mathcal{M}^\mu = 0$. This is often convenient when parametrizing loop corrections in terms of momenta, since the amplitude must satisfy Ward's identity (see e.g. the discussion at the start of P&S Sec. 7.5).

It can also be used to argue for the replacement $\sum \epsilon_\mu \epsilon_\nu^* \to -g_{\mu\nu}$ when summing over photon polarizations.

n) Why does the photon remain massless at all orders in perturbation theory, without the need for renormalization?

Solution: One way of seeing this is by the Ward identity; as argued in P&S below Eq. 7.75 this identity guarantees that the photon propagator maintains a pole at $q^2 = 0$.

A more general way of seeing this, which can also be applied in other situations, is invoking the symmetries of the Lagrangian. The QED Lagrangian is gauge invariant, meaning that it is invariant under the transformations $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$, $\psi(x) \to e^{i\alpha(x)}\psi(x)$; this is closely related to the massless nature of the photon, since a photon mass term $m^2 A_\mu A^\mu$ would not be invariant under this transformation. Thus gauge invariance guarantees a massless photon in the Lagrangian (in other words, its bare mass is zero). Since the Feynman rules, from which loop corrections are computed, are derived from the Lagrangian, Feynman diagrams to all orders must also satisfy gauge invariance. Therefore the photon will remain massless to all orders in perturbation theory, since all diagrams satisfy a symmetry that does not allow any photon mass.

o) Explain how infrared divergences occur, and how they are canceled.

Solution: Infrared (IR) divergences appear in the low-energy limit of expressions with negative mass dimension, i.e. expressions of the form $\sim 1/k^n$ (as an aside, they can usually be categorized as *collinear,* coming from the emission of a massless particle in the same direction as the original particle — or *soft,* originating from the emission of very low-energy particles).

They usually cancel when all processes that can contribute to the same observed final state are taken into account. For example, the IR divergences of $2 \to 2$ scattering at 1-loop order are canceled by adding the cross-section for $2 \to 3$ scattering in the limit where the third particle is either so low-energy or so close to one of the other final-state particles that it does not register as a separate particle in the detector.

p) Explain the main idea behind renormalization. Specialize the discussion to the case of mass and electric charge.

Solution: Ultraviolet (UV) divergences can be absorbed into the parameters in the Lagrangian by re-defining (or renormalizing) them in terms of physical observables. This exploits the fact that the parameters of the Lagrangian are not directly measurable, thus they can freely be defined to contain divergences that cancel those divergences that appear in loop integrals.

In QED, this is used to define the physical mass of the electron in terms of the pole of its propagator, and the electron charge as the strength of long-range (low-energy) electromagnetic interactions.