University of Oslo

Faculty of mathematics and natural sciences

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 1

Problem 1 Units (J. Skaar)

We will work in units where $c = 1$, $\hbar = 1$, $\varepsilon_0 = 1$. Clearly, it is trivial to transform expressions from SI units, simply by setting $c = 1$, $\hbar = 1$, $\varepsilon_0 = 1$. In the SI system,

$$c = 299792458\,\mathrm{ms}^{-1}, \tag{1a}$$

$$\hbar = 6.62607015 \times \frac{10^{-34}}{2\pi}\,\mathrm{kg\ m^2s^{-1}}, \tag{1b}$$

$$\varepsilon_0 = \frac{1}{c^2\mu_0} \approx 8.85 \times 10^{-12}\,\mathrm{kg^{-1}m^{-3}s^4A^2}. \tag{1c}$$

$$\left(\mu_0 = 4\pi \times 10^{-7}\,\mathrm{kg\,m\ s^{-2}A^{-2}}\right)$$

The inverse transformation back to SI units therefore amounts to multiplying each term in your expression by a suitable (and unique) power $c^i\hbar^j\varepsilon_0^k$ for integers i, j, k, to obtain the right SI dimension. The reason this works, is as follows. Of these three constants, ε_0 is the only one with A (ampere). Thus k is obtained uniquely. After multiplication with ε_0^k, we find j to obtain the right power of kg, before finally determining i.

For example, the fine structure constant in SI units is $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx 1/137$, which in our set of units reads $\alpha = \frac{e^2}{4\pi}$. Going back to SI units, we start by realizing that our desired result is supposed to be dimensionless. Since $\frac{e^2}{4\pi}$ has dimension $\mathrm{s^2A^2}$ in SI units, we must multiply by ε_0^{-1} to get rid of the A^2. However, then we have got a kg which was not supposed to be there. This must be canceled by multiplication by $\hbar^{-1}$. Counting the powers of m and/or s, we finally realize that we must multiply by c^{-1} to make the result dimensionless.

We use mass as the fundamental unit. We say that a quantity has dimension n if the dimension is $(\mathrm{mass})^n$.

a) Find the dimension n of mass, velocity, energy, time, length, action, Lagrangian densities, electric charge.

b) Translate the equation $\omega = m + \mathbf{p}^2/(2m)$ to SI units. Here ω is frequency, $\mathbf{p}$ is momentum, and m is mass.

c) What does the distance $r = 1/m$ correspond to? Here m is the electron mass.

Problem 2 Tensor notation (T. Bringmann, J. Van den Abeele)

This problem serves as a reminder to practice the use of tensor notation.

a) Write the following in index notation:

- ∇S (where S is a scalar).
- $\nabla \cdot \mathbf{A}$, $\nabla \times \mathbf{A}$ (where $\mathbf{A}$ is a 3D vector).
- Trace and transpose of a matrix M.

b) Prove the following 3D identities, using index notation:

- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, where $\mathbf{A}$ is a 3D vector.
- $\nabla \times (\nabla S) = 0$, where S is a scalar.

> **Hint:** First show that $A_{ab}C^{abc} = 0$ if A is symmetric in ab and C^{abc} is antisymmetric in ab.

c) Are these equalities valid? Correct where necessary!

- $\partial_\mu x^\nu = \delta_\mu^\nu$
- $\partial_\mu x^\mu = 1$
- $\partial^\mu x^\nu = g^{\mu\nu}$
- $\partial_\mu x^2 = \frac{\partial}{\partial x^\mu} x^2 = x_\mu$, where $x^2 = x_\mu x^\mu$
- $T_\alpha{}^\beta{}_\gamma = g^{\beta\mu} T_{\alpha\mu\gamma} = g^{\mu\beta} T_{\alpha\mu\gamma}$
- $T_\alpha{}^\beta{}_\beta = g_{\alpha\mu} g^{\beta\alpha} T^\mu{}_{\alpha\beta}$
- $A^\mu B_\mu = A_\mu B^\mu$
- $T_\alpha{}^\beta{}_\beta = T_{\alpha\beta}{}^\beta$

d) Construct (as many as possible)

- independent Lorentz scalars from two four-vectors A and B
- independent Lorentz scalars from a rank-2 tensor T
- independent Lorentz scalars involving one (copy of a) rank-2 tensor T and some combination of two four-vectors A and B

A scalar is here "independent" if it cannot be written as a function of the other scalars.

Problem 3 Green's function (L. L. Bratseth, J. Skaar)

Formally, a Green's function $G(x, y)$ is the inverse of a differential operator $\mathcal{D}$, in the sense that it satisfies the equation

$$\mathcal{D}G(x, y) = \delta(x - y). \tag{2}$$

In other words $G(x, y)$ is the solution to the differential equation with a forcing term given by a point source. Informally, the solution to the same differential equation with an arbitrary forcing term can be built up point by point by integrating the Green's function against the forcing term. This is equivalent to taking a superposition of solutions to the equation with point source and adding them up to the arbitrary forcing term, which is why the linearity of the differential operator is important. Formally, this means the solution to an arbitrary linear differential equation with forcing term

$$\mathcal{D}u(x) = f(x), \tag{3}$$

is given by

$$u(x) = \int d^4y\, G(x,y) f(y). \tag{4}$$

Although the resulting integrals may be difficult or impossible to compute, they provide an immediate solution to arbitrary linear differential equations when possibly no solution may be found by other methods. The solution can at the very least be computed numerically.

We restrict ourselves to translationally invariant problems, where a shift in the source $y^\mu \mapsto y^\mu + a^\mu$ leads to the same shift in the solution. Then the Green's function can be written as a function of a single spacetime coordinate:

$$\mathcal{D}G(x-y) = \delta(x-y), \tag{5}$$

or, setting $y = 0$ (putting the source in the origin),

$$\mathcal{D}G(x) = \delta(x). \tag{6}$$

In this exercise we are going to find the retarded solution of the following inhomogeneous partial differential equation, called the inhomogeneous wave equation:

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) u(\mathbf{x}, t) = f(\mathbf{x}, t). \tag{7}$$

We are interested in the retarded Green's function $G(\mathbf{x}, t)$ for this equation, obeying the conditions

$$G(\mathbf{x}, t) = 0 \quad \text{for } t < 0, \tag{8}$$

$$\lim_{|\mathbf{x}| \to \infty} G(\mathbf{x}, t) = 0, \tag{9}$$

which tells you that the source does not produce anything before it starts, and that $G(\mathbf{x}, t)$ dies far away from the source.

a) Show by the use of the Fourier transform that the Green's function in momentum-frequency space is given by

$$G(\mathbf{k}, \omega) = \frac{1}{\omega^2 - k^2}, \tag{10}$$

where $k = |\mathbf{k}|$.

b) The inverse transform of $G(\mathbf{k},\omega)$ is

$$G(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega^2 - k^2}. \tag{11}$$

Show that

$$G(\mathbf{x},t) = \frac{1}{(2\pi)^3} \frac{1}{ir} \int_{-\infty}^{\infty} dk\, k\, e^{ikr} \int d\omega \frac{e^{-i\omega t}}{\omega^2 - k^2}, \tag{12}$$

where $|\mathbf{x}| = r$. (For now we don't specify the integration path for the ω-integral; this is done in the next question.)

Hint: Note that $G(\mathbf{k},\omega)$ is spherically symmetric. Choose the coordinate system such that $\mathbf{x}$ points in the z-direction.

c) Clearly the ω-integral cannot be taken along the real axis due to the two poles in $\omega = \pm k$. Use an integration path above the poles in the complex plane (see figure p. 30 in P&S) and the residue theorem to evaluate the ω-integral, and show that

$$G(\mathbf{x},t) = -\frac{1}{4\pi|\mathbf{x}|}\delta(|\mathbf{x}| - t). \tag{13}$$

Give a physical interpretation of the result.