University of Oslo

Faculty of mathematics and natural sciences

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 1

Problem 1 Units (J. Skaar)

We will work in units where $c = 1$, $\hbar = 1$, $\varepsilon_0 = 1$. Clearly, it is trivial to transform expressions from SI units, simply by setting $c = 1$, $\hbar = 1$, $\varepsilon_0 = 1$. In the SI system,

$$c = 299792458\,\mathrm{ms}^{-1}, \tag{1a}$$

$$\hbar = 6.62607015 \times \frac{10^{-34}}{2\pi}\,\mathrm{kg\ m^2 s^{-1}}, \tag{1b}$$

$$\varepsilon_0 = \frac{1}{c^2\mu_0} \approx 8.85 \times 10^{-12}\,\mathrm{kg^{-1}m^{-3}s^4A^2}. \tag{1c}$$

$$\left(\mu_0 = 4\pi \times 10^{-7}\,\mathrm{kg\,m\ s^{-2}A^{-2}}\right)$$

The inverse transformation back to SI units therefore amounts to multiplying each term in your expression by a suitable (and unique) power $c^i\hbar^j\varepsilon_0^k$ for integers i, j, k, to obtain the right SI dimension. The reason this works, is as follows. Of these three constants, ε_0 is the only one with A (ampere). Thus k is obtained uniquely. After multiplication with ε_0^k, we find j to obtain the right power of kg, before finally determining i.

For example, the fine structure constant in SI units is $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx 1/137$, which in our set of units reads $\alpha = \frac{e^2}{4\pi}$. Going back to SI units, we start by realizing that our desired result is supposed to be dimensionless. Since $\frac{e^2}{4\pi}$ has dimension $\mathrm{s^2A^2}$ in SI units, we must multiply by ε_0^{-1} to get rid of the A^2. However, then we have got a kg which was not supposed to be there. This must be canceled by multiplication by $\hbar^{-1}$. Counting the powers of m and/or s, we finally realize that we must multiply by c^{-1} to make the result dimensionless.

We use mass as the fundamental unit. We say that a quantity has dimension n if the dimension is $(\text{mass})^n$.

a) Find the dimension n of mass, velocity, energy, time, length, action, Lagrangian densities, electric charge.

Solution: Mass clearly has $n = 1$, velocity $n = 0$ (since $c = 1$), energy $n = 1$ (consider e.g. $E = mc^2$ with $c = 1$), time $n = -1$ (recall that $\hbar\omega$ is energy, and $\hbar = 1$), length has same dimension as time since $c = 1$, action $n = 0$, Lagrangian densities $n = 4$, electric charge $n = 0$ (see expression for the fine structure constant).

b) Translate the equation $\omega = m + \mathbf{p}^2/(2m)$ to SI units. Here ω is frequency, $\mathbf{p}$ is momentum, and m is mass.

Solution: The second term on the right-hand side has dimension energy in SI units, so we translate the equation to SI units by multiplying the left-hand side by $\hbar$ and the first term on the right-hand side by c^2.

c) What does the distance $r = 1/m$ correspond to? Here m is the electron mass.

Solution: Answer in SI units: $r = \hbar/(mc)$, i.e. the Compton wavelength divided by 2π.

Problem 2 Tensor notation (T. Bringmann, J. Van den Abeele)

This problem serves as a reminder to practice the use of tensor notation.

a) Write the following in index notation:

- ∇S (where S is a scalar).
- $\nabla \cdot \mathbf{A}$, $\nabla \times \mathbf{A}$ (where $\mathbf{A}$ is a 3D vector).
- Trace and transpose of a matrix M.

Solution:

- $(\nabla S)_i = \partial_i S$
- $\nabla \cdot \mathbf{A} = \partial_i A^i$
- $(\nabla \times \mathbf{A})^i = \epsilon^{ijk} \partial_j A_k$
- $\mathrm{Tr}(M) = M_{ii}$
- $(M_{ij})^T = M_{ji}$

b) Prove the following 3D identities, using index notation:

- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, where $\mathbf{A}$ is a 3D vector.
- $\nabla \times (\nabla S) = 0$, where S is a scalar.

Hint: First show that $A_{ab}C^{abc} = 0$ if A is symmetric in ab and C^{abc} is antisymmetric in ab.

Solution: If A_{ab} is a symmetric tensor ($A_{ab} = A_{ba}$), and B antisymmetric ($B^{ab} = -B^{ba}$), then

$$A_{ab}B^{ab} = A_{ba}B^{ab} = -A_{ba}B^{ba} = -A_{ab}B^{ab}.$$

In the last equality we renamed the indices ($a \mapsto b$ and $b \mapsto a$), which of course is ok since the indices are summed over. Thus we have

$$A_{ab}B^{ab} = 0.$$

The same argument can be used to show that $A_{ab}C^{abc} = 0$ if A is symmetric in ab and C^{abc} is antisymmetric in ab (or vice versa). We can now prove the vector identities in one line:

- $\partial_i(\epsilon^{ijk}\partial_j A_k) = \epsilon^{ijk}\partial_i\partial_j A_k = 0$,
  due to symmetry $\partial_i\partial_j$ and antisymmetry of ϵ^{ijk}.
- $\epsilon^{ijk}\partial_j(\partial_k S) = 0$,
  due to symmetry of $\partial_j\partial_k$ and antisymmetry of ϵ^{ijk}.

c) Are these equalities valid? Correct where necessary!

- $\partial_\mu x^\nu = \delta^\nu_\mu$
- $\partial_\mu x^\mu = 1$
- $\partial^\mu x^\nu = g^{\mu\nu}$
- $\partial_\mu x^2 = \frac{\partial}{\partial x^\mu} x^2 = x_\mu$, where $x^2 = x_\mu x^\mu$
- $T_\alpha{}^\beta{}_\gamma = g^{\beta\mu}T_{\alpha\mu\gamma} = g^{\mu\beta}T_{\alpha\mu\gamma}$
- $T_\alpha{}^\beta{}_\beta = g_{\alpha\mu}g^{\beta\alpha}T^\mu{}_{\alpha\beta}$
- $A^\mu B_\mu = A_\mu B^\mu$
- $T_\alpha{}^\beta{}_\beta = T_{\alpha\beta}{}^\beta$

Solution:

- Right: $\partial_\mu x^\nu = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu$
- Wrong: $\partial_\mu x^\mu = \delta^\mu_\nu \partial_\mu x^\nu = \delta^\mu_\nu \delta^\nu_\mu = \delta^\mu_\mu = 4$
- Right: $\partial^\mu x^\nu = g^{\mu\rho}\partial_\rho x^\nu = g^{\mu\rho}\delta^\nu_\rho = g^{\mu\nu}$
- Wrong: Rename the summation index to avoid confusion: $x^2 = x_\alpha x^\alpha$. Then $\partial_\mu x^2 = \frac{\partial}{\partial x^\mu} x_\alpha x^\alpha = g_{\alpha\beta}\frac{\partial}{\partial x^\mu} x^\alpha x^\beta = g_{\alpha\beta}(\delta^\alpha_\mu x^\beta + x^\alpha \delta^\beta_\mu) = 2x_\mu$
- Right: using the symmetry of the metric, $g^{\mu\nu} = g^{\nu\mu}$
- Wrong: $T_\alpha{}^\beta{}_\beta = g_{\alpha\mu}g^{\beta\nu}T^\mu{}_{\nu\beta} = g^{\beta\nu}T_{\alpha\nu\beta}$
- Right: $A^\mu B_\mu = g^{\mu\alpha}A_\alpha B_\mu = A_\alpha B^\alpha$
- Right: $T_\alpha{}^\beta{}_\beta = g_{\beta\gamma}T_\alpha{}^{\beta\gamma} = T_{\alpha\gamma}{}^\gamma$

d) Construct (as many as possible)

- independent Lorentz scalars from two four-vectors A and B
- independent Lorentz scalars from a rank-2 tensor T
- independent Lorentz scalars involving one (copy of a) rank-2 tensor T and some combination of two four-vectors A and B

A scalar is here "independent" if it cannot be written as a function of the other scalars.

Solution:

- $A^2 = A^\mu A_\mu,\ B^2 = B^\mu B_\mu,\ AB = A^\mu B_\mu$ (no free indices!)
- $T^{\mu\nu}T_{\mu\nu},\ T^{\mu\nu}T_{\nu\mu},\ T^\mu{}_\mu,\ T^\mu{}_\nu T^\nu{}_\rho T^\rho{}_\mu,\ T^\mu{}_\nu T^\nu{}_\rho T^\rho{}_\lambda T^\lambda{}_\mu,\ \ldots$ (no free indices!)

  Note however that the list of *independent* scalars is still finite, as the number of components of T is finite.
- $T^{\mu\nu}A_\mu B_\nu,\ T^{\mu\nu}B_\mu A_\nu,\ T^\mu{}_\mu A^\nu B_\nu$ (no free indices!)

Problem 3 Green's function (L. L. Bratseth, J. Skaar)

Formally, a Green's function $G(x,y)$ is the inverse of a differential operator $\mathcal{D}$, in the sense that it satisfies the equation

$$\mathcal{D}G(x,y) = \delta(x-y). \tag{2}$$

In other words $G(x,y)$ is the solution to the differential equation with a forcing term given by a point source. Informally, the solution to the same differential equation with an arbitrary forcing term can be built up point by point by integrating the Green's function against the forcing term. This is equivalent to taking a superposition of solutions to the equation with point source and adding them up to the arbitrary forcing term, which is why the linearity of the differential operator is important. Formally, this means the solution to an arbitrary linear differential equation with forcing term

$$\mathcal{D}u(x) = f(x), \tag{3}$$

is given by

$$u(x) = \int d^4y\, G(x,y) f(y). \tag{4}$$

Although the resulting integrals may be difficult or impossible to compute, they provide an immediate solution to arbitrary linear differential equations when possibly no solution may be found by other methods. The solution can at the very least be computed numerically.

We restrict ourselves to translationally invariant problems, where a shift in the source $y^\mu \mapsto y^\mu + a^\mu$ leads to the same shift in the solution. Then the Green's function can be written as a function of a single spacetime coordinate:

$$\mathcal{D}G(x-y) = \delta(x-y), \tag{5}$$

or, setting $y = 0$ (putting the source in the origin),

$$\mathcal{D}G(x) = \delta(x). \tag{6}$$

In this exercise we are going to find the retarded solution of the following inhomogeneous partial differential equation, called the inhomogeneous wave equation:

$$\Big(\nabla^2 - \frac{\partial^2}{\partial t^2}\Big)u(\mathbf{x},t) = f(\mathbf{x},t). \tag{7}$$

We are interested in the retarded Green's function $G(\mathbf{x},t)$ for this equation, obeying the conditions

$$G(\mathbf{x},t) = 0 \quad \text{for } t < 0, \tag{8}$$
$$\lim_{|\mathbf{x}|\to\infty} G(\mathbf{x},t) = 0, \tag{9}$$

which tells you that the source does not produce anything before it starts, and that $G(\mathbf{x},t)$ dies far away from the source.

a) Show by the use of the Fourier transform that the Green's function in momentum-frequency space is given by

$$G(\mathbf{k},\omega) = \frac{1}{\omega^2 - k^2}, \tag{10}$$

where $k = |\mathbf{k}|$.

Solution: We look for the solution to

$$\Big(\nabla^2 - \frac{\partial^2}{\partial t^2}\Big)G(\mathbf{x},t) = \delta(\mathbf{x})\delta(t); \tag{11}$$

i.e., G is the inverse of the differential operator $(\nabla^2 - \partial_t^2)$, which is just the d'Alembertian. By Fourier transforming in space and time, we obtain

$$(-k^2 + \omega^2\Big)G(\mathbf{k},\omega) = 1, \tag{12}$$

where $G(\mathbf{k},\omega)$ is the Fourier transform of $G(\mathbf{x},t)$. Thus the Green's function in momentum-frequency space is given by

$$G(\mathbf{k},\omega) = \frac{1}{\omega^2 - k^2}. \tag{13}$$

Note that there are poles for $\omega = \pm k$.

b) The inverse transform of $G(\mathbf{k},\omega)$ is

$$G(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega^2 - k^2}. \tag{14}$$

Show that

$$G(\mathbf{x},t) = \frac{1}{(2\pi)^3}\frac{1}{ir}\int_{-\infty}^{\infty} dk\, k\, e^{ikr} \int d\omega \frac{e^{-i\omega t}}{\omega^2 - k^2}, \tag{15}$$

where $|\mathbf{x}| = r$. (For now we don't specify the integration path for the ω-integral; this is done in the next question.)

Hint: Note that $G(\mathbf{k},\omega)$ is spherically symmetric. Choose the coordinate system such that $\mathbf{x}$ points in the z-direction.

Solution: Change to polar coordinates for the integrals over $\mathbf{k}$. This allows us to deal with the $e^{i\mathbf{k}\cdot\mathbf{x}}$ factor. To this end we may choose the coordinate system such that $\mathbf{x}$ points in the z-direction:

$$\mathbf{k}\cdot\mathbf{x} = kr\cos\theta, \quad |\mathbf{k}| = k, \quad |\mathbf{x}| = r. \tag{16}$$

Then the Green's function becomes

$$G(\mathbf{x},t) = \frac{1}{(2\pi)^4}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta \int_0^{\infty} dk\,k^2 \int d\omega \frac{e^{i(kr\cos\theta-\omega t)}}{\omega^2-k^2}. \tag{17}$$

Let us first focus on the following part

$$I = \int_0^{\infty} dk\,k^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\,\frac{e^{ikr\cos\theta}}{\omega^2-k^2} \tag{18}$$

Now the ϕ integral is trivial, while the θ integral is

$$\int_0^{\pi} d\theta\,\sin\theta e^{ikr\cos\theta} = \int_{-1}^{1} du\,e^{ikru} = \frac{1}{ikr}\Big(e^{ikr}-e^{-ikr}\Big). \tag{19}$$

Then

$$\begin{aligned}
I &= \frac{2\pi}{ir}\int_0^{\infty} dk\,\frac{k}{\omega^2-k^2}\Big(e^{ikr}-e^{-ikr}\Big)\\
&= \frac{2\pi}{ir}\Big(\int_0^{\infty} dk\frac{k}{\omega^2-k^2}e^{ikr} + \int_{\infty}^{0} dk\frac{k}{\omega^2-k^2}e^{-ikr}\Big)\\
&= \frac{2\pi}{ir}\Big(\int_0^{\infty} dk\frac{k}{\omega^2-k^2}e^{ikr} + \int_{-\infty}^{0} dk\frac{k}{\omega^2-k^2}e^{ikr}\Big)\\
&= \frac{2\pi}{ir}\int_{-\infty}^{\infty} dk\frac{k}{\omega^2-k^2}e^{ikr}
\end{aligned} \tag{20}$$

where we in the second step sent $k\to -k$ in the second integral. Plugging this back into the Green's function we obtain

$$G(\mathbf{x},t) = \frac{1}{(2\pi)^3}\frac{1}{ir}\int_{-\infty}^{\infty} dk\,k\,e^{ikr}\int d\omega\frac{e^{-i\omega t}}{\omega^2-k^2}. \tag{21}$$

c) Clearly the ω-integral cannot be taken along the real axis due to the two poles in $\omega = \pm k$. Use an integration path above the poles in the complex plane (see figure p. 30 in P&S) and the residue theorem to evaluate the ω-integral, and show that

$$G(\mathbf{x},t) = -\frac{1}{4\pi|\mathbf{x}|}\delta(|\mathbf{x}|-t). \tag{22}$$

Give a physical interpretation of the result.

Solution: First consider $t > 0$. Since $e^{-i\omega t}$ blows up in the upper half-plane, we close the contour in the lower half plane. Then, the integration along the half-circle tends to zero as the radius $R \to \infty$:

$$\left| \int_{\text{half circle}} d\omega \frac{e^{-i\omega t}}{\omega^2 - k^2} \right| \le \int_{\text{half circle}} \left| d\omega \frac{1}{\omega^2 - k^2} \right| \le \pi R \frac{1}{R^2 - k^2} \to 0.$$

Both poles (at $\omega = \pm k$) are enclosed by the contour, and the residue theorem gives

$$\int d\omega \frac{e^{-i\omega t}}{(\omega - k)(\omega + k)} = -2\pi i \frac{e^{-ikt}}{2k} + 2\pi i \frac{e^{ikt}}{2k}. \tag{23}$$

Inserting this result back into (21), we obtain

$$G(\mathbf{x}, t) = -\frac{1}{4\pi r} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(r-t)} + \frac{1}{4\pi r} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(r+t)} = -\frac{1}{4\pi r} \delta(r - t), \tag{24}$$

where we have used $t > 0$ in the last equality. For $t < 0$ we close the contour in the upper half-plane, which leads to the result $G(\mathbf{x}, t) = 0$.

The Green's function is the response to a point source in the origin with an impulse at $t = 0$. The source leads to a short spherical wave pulse propagating in the $+r$-direction, but decaying as $1/r$. This is similar to throwing a stone in the water and observing the circular wave pulse propagating away.

The Green's functions we will encounter in the course has the fitting name *propagators*.

For the interested student: One may ask why the chosen integration path is the correct path for the inverse transform, given requirement (8). We do the analysis more rigorously as follows.

The appearance of poles at the real ω-axis shows that the Fourier transform of $G(\mathbf{x}, t)$ does not exist everywhere. We therefore define $H(\mathbf{x}, t) = G(\mathbf{x}, t)e^{-\epsilon t}$ (for $\epsilon > 0$), which clearly is Fourier transformable. [Recall the assumption $G(\mathbf{x}, t) = 0$ for $t < 0$ (which must be verified in the end).] We have

$$G(\mathbf{x}, t) = H(\mathbf{x}, t)e^{\epsilon t}, \tag{25}$$

and therefore

$$\partial_t^2 G(\mathbf{x}, t) = e^{\epsilon t} \left(\partial_t^2 + 2\epsilon \partial_t + \epsilon^2 \right) H(\mathbf{x}, t). \tag{26}$$

Our differential equation (11) now becomes

$$e^{\epsilon t} \left(\nabla^2 - \partial_t^2 - 2\epsilon \partial_t - \epsilon^2 \right) H(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t). \tag{27}$$

Multiplication by $e^{-\epsilon t}$ on both sides gives

$$\left(\nabla^2 - \partial_t^2 - 2\epsilon \partial_t - \epsilon^2 \right) H(\mathbf{x}, t) = \delta(\mathbf{x})\delta(t). \tag{28}$$

Since the Fourier transform of H (and the derivatives) exists, we may Fourier transform the equation:

$$\left(-k^2 + \omega^2 + 2i\omega\epsilon - \epsilon^2 \right) H(\mathbf{k}, \omega) = 1. \tag{29}$$

The solution is

$$H(\mathbf{k}, \omega) = \frac{1}{(\omega + i\epsilon)^2 - k^2}, \tag{30}$$

with poles in $\omega = \pm k - i\epsilon$.

The inverse Fourier transform involves integration along the real ω-axis. For $t > 0$ we close the contour in the lower half-plane, picking up both residues. This shows that we used the correct integration path above. For $t < 0$ we must close the contour in the upper half-plane, leading to $H(\mathbf{x}, t) = 0$ and therefore $G(\mathbf{x}, t) = 0$ as before.

It is interesting to note that we did not have to take the limit $\epsilon \to 0$; the result is obtained using any $\epsilon > 0$.

An equivalent and simpler argument, is to use the Laplace transform in time rather than the Fourier transform. The analysis can then be done directly on G (we don't need H). After finding the solution in the Laplace transform domain, the solution in time, for $t > 0$, is found by the inverse Laplace transform. Since the inverse Laplace transform amounts to including both residues, we get the same result as above.