University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 2

Problem 1 Two scalar fields (T. Bringmann)

Consider two interacting, real, scalar fields ϕ_1 and ϕ_2, described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}m_2^2\phi_2^2 - \lambda\phi_1^2\phi_2^2 \,, \tag{1}$$

where λ is a constant.

Derive the equations of motion for the theory (1). Do so by directly using the principle of least action (by varying the fields), i.e., not by using the Euler-Lagrange equations. Give an explanation of why the boundary terms do not contribute in your result, and discuss the relation to the Klein-Gordon-equation(s) you would expect for non-interacting fields.

Problem 2 Obtaining ladder operators from fields (J. Skaar)

a) The hermitian Klein-Gordon field $\phi(x)$ and its canonical momentum density $\pi(x)$ are

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right) , \tag{2a}$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{-ipx} - a_{\mathbf{p}}^\dagger e^{ipx} \right) , \tag{2b}$$

where $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. Find a method to invert these relations; thus obtaining the ladder operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ from $\phi(x)$ and $\pi(x)$.

> **Hint:** Set $t = x^0 = 0$, and try to write the expressions as Fourier transforms by changing integration variable in one of the terms for each field.

b) By requiring equal-time commutators $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$, show that we must have

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta(\mathbf{p} - \mathbf{q}), \tag{3a}$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \tag{3b}$$

> **Hint:** Follows from the result in a), after determining $[\phi(\mathbf{p}), \pi^\dagger(\mathbf{q})]$ and $[\phi(\mathbf{p}), \pi(\mathbf{q})]$ (the commutator of the Fourier transformed fields).

c) Prove that

$$a_{\mathbf{p}} = i \int d^3x \Psi^*_{\mathbf{p}}(x) \overleftrightarrow{\partial}_0 \phi(x) \tag{4}$$

for $x^0 = 0$. Here we have defined

$$\Psi_{\mathbf{p}}(x) = \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx}, \quad p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \tag{5}$$

and the symbol $\overleftrightarrow{\partial}_0$, which operates "in both directions":

$$f \overleftrightarrow{\partial}_0 g = f \partial_0 g - (\partial_0 f) g. \tag{6}$$

We will use expression (4) later in the course, when discussing the important LSZ formula.

> **Hint:** Substitute (5) and compare to the result in a).

Problem 3 The complex Klein-Gordon field (Peskin & Schroeder exercise 2.2, modified)

This problem considers the quantization of a *complex*, scalar field. It is both a good recap on how we quantized the real scalar field, and at the same time needs concepts that we meet when quantizing fermions (in particular the necessary appearance of anti-particles). The Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi.$$

It is easiest to analyze this theory by considering ϕ and ϕ^* (rather than the real and imaginary parts) as the independent, dynamical variables.

a) Show that the Hamilton operator is given by

$$H = \int d^3x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right)$$

b) Compared to the real case, the theory exhibits an additional symmetry, $\phi \mapsto e^{-i\alpha}\phi$. Show that this is indeed a symmetry, and user Noether's theorem to show that the corresponding conserved charge is proportional to

$$Q = -i \int d^3x (\dot{\phi}^* \phi - \phi^* \dot{\phi}). \tag{7}$$

> **Hint:** A similar calculation has already been done in the lectures.

So far we have considered the classical fields ϕ and ϕ^*. We now quantize the theory, by promoting the fields to operators ϕ and $\phi^\dagger$, respectively. Similarly, the canonical momentum densities are π and $\pi^\dagger$. Complex conjugation ($*$) in the classical expressions above must therefore be replaced by hermitian conjugate ($\dagger$).

We require

$$
\begin{aligned}
[\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \\
[\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}),
\end{aligned} \tag{8}
$$

while

$$
\begin{aligned}
[\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})], \\
[\phi(\mathbf{x}), \phi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi^\dagger(\mathbf{y})], \\
[\phi(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \phi^\dagger(\mathbf{y})].
\end{aligned} \tag{9}
$$

c) In the Heisenberg picture the fields evolve according to the Heisenberg equations of motion,

$$
i\frac{\partial \phi(x)}{\partial t} = [\phi(x), H] \tag{10a}
$$

$$
i\frac{\partial \pi(x)}{\partial t} = [\pi(x), H]. \tag{10b}
$$

Prove that ϕ and π satisfy the following Hamilton's equations of motion:

$$
\frac{\partial \phi(x)}{\partial t} = \pi^\dagger(x), \tag{11a}
$$

$$
\frac{\partial \pi(x)}{\partial t} = (\nabla^2 - m^2)\phi^\dagger(x). \tag{11b}
$$

From these deduce that ϕ satisfies the Klein-Gordon equation.

d) Introduce annihilation and creation operators, and show that

$$
H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}) \tag{12}
$$

if we ignore an infinite constant (as before). Here you should convince yourself that the Fourier decomposition of a complex function $\phi(\mathbf{x})$, already at the classical level, requires twice as many independent Fourier coefficients as for a real function. When promoted to operators, you should thus denote those with, e.g., $a_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$ (rather than $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$).

Argue that the theory contains two types of particles with mass m.

e) Rewrite the conserved charge (7) in terms of the annihilation and creation operators introduced above. How does the charge between the two types of particles differ?

Problem 4 Complex fields (optional; for the interested student)

(J. Skaar)

In the lectures, and above, we have treated ϕ and ϕ^* as independent. We will now justify this method, which is far from obvious, since ϕ^* in fact is uniquely determined by ϕ.

We write

$$
\phi = \phi_r + i\phi_i, \tag{13}
$$

where ϕ_r and ϕ_i are the real and imaginary parts of ϕ, respectively. Clearly, we may treat ϕ_r and ϕ_i as independent fields, that can be varied separately to obtain Euler-Lagrange equations for ϕ_r and ϕ_i.

Instead of ϕ_r and ϕ_i as independent "coordinates" in the complex ϕ-plane, let us fix ϕ_0, and use real parameters s and t to express

$$\phi_r = \frac{1}{2}[s\phi_0 + t\phi_0^*]\,, \tag{14a}$$
$$\phi_i = \frac{1}{2i}[s\phi_0 - t\phi_0^*]\,, \tag{14b}$$

or

$$s\phi_0 = \phi_r + i\phi_i, \tag{15a}$$
$$t\phi_0^* = \phi_r - i\phi_i, \tag{15b}$$

This gives

$$\frac{1}{\phi_0}\frac{\partial}{\partial s} = \frac{\partial}{\partial(s\phi_0)} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} - i\frac{\partial}{\partial\phi_i}\right], \tag{16a}$$
$$\frac{1}{\phi_0^*}\frac{\partial}{\partial t} = \frac{\partial}{\partial(t\phi_0^*)} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} + i\frac{\partial}{\partial\phi_i}\right]. \tag{16b}$$

We therefore write

$$\frac{\partial}{\partial\phi} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} - i\frac{\partial}{\partial\phi_i}\right], \tag{17a}$$
$$\frac{\partial}{\partial\phi^*} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} + i\frac{\partial}{\partial\phi_i}\right], \tag{17b}$$

but have in mind that they actually mean (16). For example, $\frac{\partial}{\partial\phi}$ means that one varies $s\phi_0$ while holding $t\phi_0^*$ constant.

a) Prove that $\frac{\partial\phi}{\partial\phi} = \frac{\partial\phi^*}{\partial\phi^*} = 1$ and $\frac{\partial\phi}{\partial\phi^*} = \frac{\partial\phi^*}{\partial\phi} = 0$.

b) Argue that the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} = 0, \tag{18a}$$
$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0, \tag{18b}$$

imply

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0, \tag{19a}$$
$$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = 0. \tag{19b}$$

Hint: Consider $\frac{1}{2}$[(18a) $-$ i(18b)]. Note that (17) can be applied to identify what $\frac{\partial}{\partial(\partial_\mu\phi)}$ means, by thinking of $(\partial_\mu\phi)$ as the complex field rather than ϕ.

c) Go through the derivation of Noether's theorem, and convince yourself that the method of treating ϕ and ϕ^* as independent fields gives the same result as if you consider ϕ_r and ϕ_i as the independent fields.

Hint: From the above, we know that the Euler-Lagrange equations for ϕ and ϕ^* are satisfied (19). Following the usual proof, assuming that ϕ and ϕ^* are independent fields we can express the change in the Lagrangian as

$$\Delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta\phi^* \right). \tag{20}$$

Show that (20) can be rewritten to

$$\Delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Delta\phi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Delta\phi_i \right). \tag{21}$$