# University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 2

## Problem 1 Two scalar fields (T. Bringmann)

Consider two interacting, real, scalar fields $\phi_1$ and $\phi_2$, described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) - \frac{1}{2} m_1^2 \phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2} m_2^2 \phi_2^2 - \lambda \phi_1^2 \phi_2^2 \,, \quad (1)$$

where $\lambda$ is a constant.

Derive the equations of motion for the theory (1). Do so by directly using the principle of least action (by varying the fields), i.e., not by using the Euler-Lagrange equations. Give an explanation of why the boundary terms do not contribute in your result, and discuss the relation to the Klein-Gordon-equation(s) you would expect for non-interacting fields.

**Solution:** We consider the two real scalar fields $\phi_i$ ($i = 1, 2$) as independent. Under a change of one of them, say $\phi_1 \to \phi_1 + \delta\phi_1$, the Lagrangian changes as $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$ with

$$\delta\mathcal{L} = (\partial_\mu \phi_1)(\partial^\mu \delta\phi_1) - m_1^2 \phi_1 \delta\phi_1 - 2\lambda \phi_1 \phi_2^2 \delta\phi_1,$$

where we used the fact that $\delta(\partial_\mu \phi_1) = \partial_\mu(\phi_1 + \delta\phi_1) - \partial_\mu(\phi_1) = \partial_\mu(\delta\phi_1)$. The first term can be rewritten to $\partial^\mu[(\partial_\mu \phi_1)\delta\phi_1] - \Box\phi_1(\delta\phi_1)$. After substitution into the action integral $\delta S = \int d^4x \, \delta\mathcal{L}$, the term $\partial^\mu[(\partial_\mu \phi_1)\delta\phi_1]$ leads to a surface integral of $\partial_\mu \phi_1(\delta\phi_1)$, by the divergence theorem in four dimensions. Thus if we restrict ourselves to variations $\delta\phi_i$ that vanish at the boundaries, we can ignore the term.

This gives

$$\delta S = -\int d^4x [\Box\phi_1 + m_1^2 \phi_1 + 2\lambda \phi_1 \phi_2^2] \delta\phi_1.$$

Requiring $\delta S$ to vanish for an arbitrary variation $\delta\phi_1$ leads to the equation of motion (EOM) for $\phi_1$:

$$\Box\phi_1 + m_1^2 \phi_1 + 2\lambda \phi_1 \phi_2^2 = 0.$$

As the Lagrangian is symmetric under the interchange of field labels $1 \leftrightarrow 2$, the EOM for $\phi_2$ immediately follows:

$$\Box\phi_2 + m_2^2 \phi_2 + 2\lambda \phi_1^2 \phi_2 = 0.$$

The Lagrangian term $-\lambda \phi_1^2 \phi_2^2$ couples the two fields, leading to $\phi_2$ showing up in the EOM for $\phi_1$ and vice versa, meaning that they interact. Klein-Gordon equations for both fields are retrieved in the limit where the coupling vanishes ($\lambda \to 0$).

## Problem 2 Obtaining ladder operators from fields

**(J. Skaar)**

**a)** The hermitian Klein-Gordon field $\phi(x)$ and its canonical momentum density $\pi(x)$ are

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right), \tag{2a}$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left( a_{\mathbf{p}} e^{-ipx} - a_{\mathbf{p}}^\dagger e^{ipx} \right), \tag{2b}$$

where $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. Find a method to invert these relations; thus obtaining the ladder operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ from $\phi(x)$ and $\pi(x)$.

**Hint:** Set $t = x^0 = 0$, and try to write the expressions as Fourier transforms by changing integration variable in one of the terms for each field.

**Solution:** We set $t = x^0 = 0$, change integration variable $\mathbf{p} \mapsto -\mathbf{p}$ in the last terms, and use $E_{-\mathbf{p}} = E_{\mathbf{p}}$:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{3a}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left( a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{3b}$$

If $\phi(\mathbf{p})$ and $\pi(\mathbf{p})$ are the Fourier transforms of $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$, respectively, (3) shows that

$$\phi(\mathbf{p}) = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right), \tag{4a}$$

$$\boldsymbol{\pi}(\mathbf{p}) = -i \sqrt{\frac{E_{\mathbf{p}}}{2}} \left( a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger \right), \tag{4b}$$

or

$$a_{\mathbf{p}} = \sqrt{\frac{E_{\mathbf{p}}}{2}} \phi(\mathbf{p}) + \frac{i}{\sqrt{2E_{\mathbf{p}}}} \pi(\mathbf{p}), \tag{5a}$$

$$a_{\mathbf{p}}^\dagger = \sqrt{\frac{E_{\mathbf{p}}}{2}} \phi^\dagger(\mathbf{p}) - \frac{i}{\sqrt{2E_{\mathbf{p}}}} \pi^\dagger(\mathbf{p}). \tag{5b}$$

Thus we can find the ladder operators from the fields by setting $x^0 = 0$, Fourier transform them, and plugging into (5). (Note that even though $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are hermitian (corrsponding to real, classical fields), the Fourier transformed fields $\phi(\mathbf{p})$ and $\pi(\mathbf{p})$ are not necessarily hermitian.)

**b)** By requiring equal-time commutators $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$, show that we must have

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta(\mathbf{p} - \mathbf{q}), \tag{6a}$$
$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \tag{6b}$$

> **Hint:** Follows from the result in a), after determining $[\phi(\mathbf{p}), \pi^\dagger(\mathbf{q})]$ and $[\phi(\mathbf{p}), \pi(\mathbf{q})]$ (the commutator of the Fourier transformed fields).

**c)** Prove that

$$a_{\mathbf{p}} = i \int d^3x \Psi_{\mathbf{p}}^*(x) \overleftrightarrow{\partial}_0 \phi(x) \tag{7}$$

for $x^0 = 0$. Here we have defined

$$\Psi_{\mathbf{p}}(x) = \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx}, \quad p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \tag{8}$$

and the symbol $\overleftrightarrow{\partial}_0$, which operates "in both directions":

$$f \overleftrightarrow{\partial}_0 g = f \partial_0 g - (\partial_0 f) g. \tag{9}$$

We will use expression (7) later in the course, when discussing the important LSZ formula.

> **Hint:** Substitute (8) and compare to the result in a).

## Problem 3 The complex Klein-Gordon field (Peskin & Schroeder exercise 2.2, modified)

This problem considers the quantization of a *complex*, scalar field. It is both a good recap on how we quantized the real scalar field, and at the same time needs concepts that we meet when quantizing fermions (in particular the necessary appearance of anti-particles). The Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi.$$

It is easiest to analyze this theory by considering $\phi$ and $\phi^*$ (rather than the real and imaginary parts) as the independent, dynamical variables.

**a)** Show that the Hamilton operator is given by

$$H = \int d^3x \left( \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right)$$

**Solution:** Starting with the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi = \partial_0 \phi^* \partial^0 \phi - \nabla\phi^* \cdot \nabla\phi - m^2 \phi^* \phi, \tag{10}$$

we find the conjugate momenta:

$$\begin{aligned} \pi &= \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \partial_0 \phi^* \\ \pi^* &= \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^*} = \partial_0 \phi \end{aligned} \tag{11}$$

From these expressions we can calculate the Hamiltonian density:

$$\begin{aligned} \mathcal{H} &= \Sigma \pi_i \dot{\phi}_i - \mathcal{L} \\ &= \pi\pi^* + \pi^*\pi - (\pi\pi^* - \nabla\phi^* \cdot \nabla\phi - m^2\phi^*\phi) \\ &= \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi, \end{aligned} \tag{12}$$

and thus the Hamiltonian is $H = \int d^3\mathbf{x}\, (\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi)$.

**b)** Compared to the real case, the theory exhibits an additional symmetry, $\phi \mapsto e^{-i\alpha}\phi$. Show that this is indeed a symmetry, and user Noether's theorem to show that the corresponding conserved charge is proportional to

$$Q = -i \int d^3x (\dot{\phi}^* \phi - \phi^* \dot{\phi}). \tag{13}$$

**Hint:** A similar calculation has already been done in the lectures.

So far we have considered the classical fields $\phi$ and $\phi^*$. We now quantize the theory, by promoting the fields to operators $\phi$ and $\phi^\dagger$, respectively. Similarly, the canonical momentum densities are $\pi$ and $\pi^\dagger$. Complex conjugation $(*)$ in the classical expressions above must therefore be replaced by hermitian conjugate $(\dagger)$.

We require

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \\ [\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \end{aligned} \tag{14}$$

while

$$\begin{aligned} [\phi(\mathbf{x}), \phi(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})], \\ [\phi(\mathbf{x}), \phi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \pi^\dagger(\mathbf{y})], \\ [\phi(\mathbf{x}), \pi^\dagger(\mathbf{y})] &= 0 = [\pi(\mathbf{x}), \phi^\dagger(\mathbf{y})]. \end{aligned} \tag{15}$$

**c)** In the Heisenberg picture the fields evolve according to the Heisenberg equations of motion,

$$i\frac{\partial \phi(x)}{\partial t} = [\phi(x), H] \tag{16a}$$

$$i\frac{\partial \pi(x)}{\partial t} = [\pi(x), H]. \tag{16b}$$

Prove that $\phi$ and $\pi$ satisfy the following Hamilton's equations of motion:

$$\frac{\partial \phi(x)}{\partial t} = \pi^\dagger(x), \tag{17a}$$
$$\frac{\partial \pi(x)}{\partial t} = (\nabla^2 - m^2)\phi^\dagger(x). \tag{17b}$$

From these deduce that $\phi$ satisfies the Klein-Gordon equation.

**Solution:**

$$\begin{aligned} i\frac{\partial \phi(x)}{\partial t} = [\phi(x), H] &= \left[\phi(x), \int d^3\mathbf{y}\,(\pi^\dagger\pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi)\right] \\ &= \int d^3\mathbf{y}\,[\phi(x), \pi(y)]\pi^\dagger(y) \\ &= \int d^3\mathbf{y}\,(i\delta^{(3)}(\mathbf{x}-\mathbf{y}))\pi^\dagger(y) \\ &= i\pi^\dagger(x). \end{aligned} \tag{18}$$

Here we used the fact that $\phi$ commute with all terms in $\mathcal{H}$, except $\pi$. To derive the next Hamilton equation, it is useful to note that

$$H = \int d^3\mathbf{y}\,(\pi^\dagger\pi + \nabla(\phi\nabla\phi^\dagger) - \phi\nabla^2\phi^\dagger + m^2\phi^\dagger\phi). \tag{19}$$

Here the second term can be written as a surface integral of $\phi\nabla\phi^\dagger$. After substitution into

$$i\frac{\partial \pi(x)}{\partial t} = [\pi(x), H] \tag{20}$$

the first two terms of $H$ can be ignored; the first because $\pi$ commutes with $\pi^\dagger\pi$, and the second because we may take the surface integral sufficiently far out, away from $\mathbf{x}$, which ensures that the commutator vanishes. The last two terms lead to terms proportional to a delta function, which after integration give

$$i\frac{\partial \pi(x)}{\partial t} = i(\nabla^2\phi^\dagger - m^2\phi^\dagger). \tag{21}$$

By combining (18) and (21) we obtain the Klein-Gordon equation of motion:

$$\frac{\partial^2 \phi(x)}{dt^2} - \nabla^2\phi(x) + m^2\phi(x) = 0. \tag{22}$$

**d)** Introduce annihilation and creation operators, and show that

$$H = \int \frac{d^3p}{(2\pi)^3} E_\mathbf{p}(a_\mathbf{p}^\dagger a_\mathbf{p} + b_\mathbf{p}^\dagger b_\mathbf{p}) \tag{23}$$

if we ignore an infinite constant (as before). Here you should convince yourself that the Fourier decomposition of a complex function $\phi(\mathbf{x})$, already at the classical level, requires twice as many independent Fourier coefficients as for a real function. When promoted to operators, you should thus denote those with, e.g., $a_\mathbf{k}$ and $b_\mathbf{k}^\dagger$ (rather than $a_\mathbf{k}$ and $a_\mathbf{k}^\dagger$).

Argue that the theory contains two types of particles with mass $m$.

**Solution:** We start by introducing creation and annihilation operators for the field $\phi$, noting now that $\phi$ is no longer hermitian ($\phi \neq \phi^\dagger$). Therefore, we cannot assume that the coefficient of $e^{ip\cdot x}$ is just the adjoint of that of $e^{-ip\cdot x}$. We are thus forced to introduce two sets of operators, $a_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$:

$$\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ip\cdot x} + b^\dagger_{\mathbf{p}}e^{ip\cdot x})\Big|_{p^0=E_{\mathbf{p}}} \\
\phi^\dagger(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(a^\dagger_{\mathbf{p}}e^{ip\cdot x} + b_{\mathbf{p}}e^{-ip\cdot x})\Big|_{p^0=E_{\mathbf{p}}} \\
\pi(x) = \dot{\phi}^\dagger(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(iE_{\mathbf{p}}a^\dagger_{\mathbf{p}}e^{ip\cdot x} - iE_{\mathbf{p}}b_{\mathbf{p}}e^{-ip\cdot x})\Big|_{p^0=E_{\mathbf{p}}} \\
\pi^\dagger(x) = \dot{\phi}(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(-iE_{\mathbf{p}}a_{\mathbf{p}}e^{-ip\cdot x} + iE_{\mathbf{p}}b^\dagger_{\mathbf{p}}e^{ip\cdot x})\Big|_{p^0=E_{\mathbf{p}}}
\end{aligned} \tag{24}$$

The Hamiltonian must be independent of time (why?), so we set $x^0 = 0$. Plugging into the Hamiltonian,

$$\begin{aligned}
H &= \int d^3\mathbf{x}(\pi^\dagger\pi + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi) \\
&= \int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(-iE_{\mathbf{p}}a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + iE_{\mathbf{p}}b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}) \\
&\cdot \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}(iE_{\mathbf{q}}a^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}} - iE_{\mathbf{q}}b_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}) \\
&+ \int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(i\mathbf{p}a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - i\mathbf{p}b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}) \\
&\cdot \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}(-i\mathbf{q}a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} + i\mathbf{q}b^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}) \\
&+ m^2\int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}(a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}) \\
&\cdot \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} + b^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}).
\end{aligned} \tag{25}$$

Performing the $\int d^3\mathbf{x}$ integral yields a delta function, which collapses the integration with respect to $\mathbf{q}$. After a little work this leads to

$$H = \int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}}) = \int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}). \tag{26}$$

For the last equality, we dropped an infinite constant. In such calculations it is also useful to note that $E_{-\mathbf{p}} = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, and it may be helpful to use the substitution $-\mathbf{p} \mapsto \mathbf{p}$ in some integrals.

The final expression is exactly as for the real (or hermitian) Klein-Gordon field, except that now there are two (number operator) terms $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}b_{\mathbf{p}}$. As observables these count the number of "a" and "b" particles in mode $\mathbf{p}$.

From (14) and (15) it is possible to obtain the usual ladder operator commutators for $a_{\mathbf{p}}$, and for $b_{\mathbf{p}}$, and that all commutators between

$a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ operators vanish. Thus we must consider the "a" and "b" excitations as different particles.

**e)** Rewrite the conserved charge (13) in terms of the annihilation and creation operators introduced above. How does the charge between the two types of particles differ?

**Solution:** Starting from the expression found in b), plugging in field expressions and using commutation relations, we find after some work

$$Q = \int \frac{d^3p}{(2\pi)^3}(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}}). \tag{27}$$

Note how particles created by the $a^\dagger$ and $b^\dagger$ operators get opposite charge!

## Problem 4 Complex fields (optional; for the interested student)

**(J. Skaar)**

In the lectures, and above, we have treated $\phi$ and $\phi^*$ as independent. We will now justify this method, which is far from obvious, since $\phi^*$ in fact is uniquely determined by $\phi$.

We write

$$\phi = \phi_r + i\phi_i, \tag{28}$$

where $\phi_r$ and $\phi_i$ are the real and imaginary parts of $\phi$, respectively. Clearly, we may treat $\phi_r$ and $\phi_i$ as independent fields, that can be varied separately to obtain Euler-Lagrange equations for $\phi_r$ and $\phi_i$.

Instead of $\phi_r$ and $\phi_i$ as independent "coordinates" in the complex $\phi$-plane, let us fix $\phi_0$, and use real parameters $s$ and $t$ to express

$$\phi_r = \frac{1}{2}[s\phi_0 + t\phi_0^*]\,, \tag{29a}$$

$$\phi_i = \frac{1}{2i}[s\phi_0 - t\phi_0^*]\,, \tag{29b}$$

or

$$s\phi_0 = \phi_r + i\phi_i, \tag{30a}$$

$$t\phi_0^* = \phi_r - i\phi_i, \tag{30b}$$

This gives

$$\frac{1}{\phi_0}\frac{\partial}{\partial s} = \frac{\partial}{\partial(s\phi_0)} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} - i\frac{\partial}{\partial\phi_i}\right], \tag{31a}$$

$$\frac{1}{\phi_0^*}\frac{\partial}{\partial t} = \frac{\partial}{\partial(t\phi_0^*)} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} + i\frac{\partial}{\partial\phi_i}\right]. \tag{31b}$$

We therefore write

$$\frac{\partial}{\partial\phi} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} - i\frac{\partial}{\partial\phi_i}\right], \tag{32a}$$

$$\frac{\partial}{\partial\phi^*} = \frac{1}{2}\left[\frac{\partial}{\partial\phi_r} + i\frac{\partial}{\partial\phi_i}\right], \tag{32b}$$

but have in mind that they actually mean (31). For example, $\frac{\partial}{\partial\phi}$ means that one varies $s\phi_0$ while holding $t\phi_0^*$ constant.

**a)** Prove that $\frac{\partial\phi}{\partial\phi} = \frac{\partial\phi^*}{\partial\phi^*} = 1$ and $\frac{\partial\phi}{\partial\phi^*} = \frac{\partial\phi^*}{\partial\phi} = 0$.

**b)** Argue that the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)} = 0, \tag{33a}$$

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0, \tag{33b}$$

imply

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0, \tag{34a}$$

$$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = 0. \tag{34b}$$

**Hint:** Consider $\frac{1}{2}$[(33a) $-$ $i$(33b)]. Note that (32) can be applied to identify what $\frac{\partial}{\partial(\partial_\mu\phi)}$ means, by thinking of $(\partial_\mu\phi)$ as the complex field rather than $\phi$.

**c)** Go through the derivation of Noether's theorem, and convince yourself that the method of treating $\phi$ and $\phi^*$ as independent fields gives the same result as if you consider $\phi_r$ and $\phi_i$ as the independent fields.

**Hint:** From the above, we know that the Euler-Lagrange equations for $\phi$ and $\phi^*$ are satisfied (34). Following the usual proof, assuming that $\phi$ and $\phi^*$ are independent fields we can express the change in the Lagrangian as

$$\Delta\mathcal{L} = \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^* \right). \tag{35}$$

Show that (35) can be rewritten to

$$\Delta\mathcal{L} = \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_r)}\Delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\Delta\phi_i \right). \tag{36}$$