University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 3

Problem 1 Single particles in wave packets (J. Skaar)

In this problem, you will construct single particles in wave packets and view their propagation in space. The nonrelativistic and ultrarelativistic limits will be examined. A final goal is to understand the definition of the S-matrix.

Note that usually p^2 means $p^2 = (p^0)^2 - \mathbf{p}^2$, but when we limit ourselves to one spatial dimension, p may refer to one-dimensional momentum.

a) Our particle will be expressed as

$$|\psi\rangle = a_\psi^\dagger |0\rangle, \quad a_\psi^\dagger = \int \frac{d^3p}{(2\pi)^3}\psi(\mathbf{p})a_\mathbf{p}^\dagger, \tag{1}$$

where $a_\mathbf{p}^\dagger$ is the usual creation operator for the mode with momentum $\mathbf{p}$. Here $a_\psi^\dagger$ should be viewed as a wavepacket creation operator. We let the wavepacket spectrum $\psi(\mathbf{p})$ be normalized by

$$\int \frac{d^3p}{(2\pi)^3}|\psi(\mathbf{p})|^2 = 1.$$

Calculate $[a_\psi, a_\psi^\dagger]$, $\langle\psi|\psi\rangle$, and $[a_\mathbf{p}, a_\psi^\dagger]$.

b) One observable is the field operator itself,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ipx} + a_\mathbf{p}^\dagger e^{ipx}\right), \quad p^0 = E_\mathbf{p} = \sqrt{\mathbf{p}^2+m^2}. \tag{2}$$

Show that $\langle\psi|\phi(x)|\psi\rangle = 0$. This does not mean that a measurement of the field in this state always gives zero; it may fluctuate around zero with zero expectation value. Indeed, prove that the expectation value of the normal-ordered ($\mathcal{N}$) observable $\phi^2(x)$ is

$$\langle\psi|\mathcal{N}\{\phi^2(x)\}|\psi\rangle = 2|\Psi(x)|^2, \tag{3}$$

where

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{\psi(\mathbf{p})}{\sqrt{2E_\mathbf{p}}}e^{-ipx}. \tag{4}$$

Normal ordering means to put all creation operators to the left, i.e., $\mathcal{N}(aa^\dagger) = a^\dagger a$ and $\mathcal{N}(a^\dagger a) = a^\dagger a$.

Note: The free field operator (2) *only contains a standard, plane-wave x-dependence. Thus, although the Heisenberg picture state $|\psi\rangle$ is independent of space and time, the actual space and time dependence of the particle is encoded into $|\psi\rangle$!*

Hint: Substitute the field expression in $\langle 0|a_\psi\mathcal{N}\{\phi^2(x)\}a_\psi^\dagger|0\rangle$. Use the commutator $[a_\mathbf{p}, a_\psi^\dagger] = \psi(\mathbf{p})$.

c) We will now consider the expectation value $|\Psi(x)|^2$ in more detail, thinking of it as the "shape of the particle". Argue that $\Psi(\mathbf{x}, t = 0)$ can have any shape in space, by tailoring the wavepacket spectrum $\psi(\mathbf{p})$. Choose $\psi(\mathbf{p})/\sqrt{2E_\mathbf{p}}$ such that $|\Psi(\mathbf{x}, t = 0)|^2$ is a 3d gaussian centered at the point $\mathbf{b}$.

d) Peskin & Schroeder (p. 24) claims that the state

$$\phi(\mathbf{b})\,|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{b}}}{\sqrt{2E_\mathbf{p}}} a^\dagger_\mathbf{p}\,|0\rangle$$

describes a particle at location $\mathbf{b}$. (See (2.41) and (2.35).) Here $\phi(\mathbf{b})$ denotes the field operator at position $\mathbf{x} = \mathbf{b}$ and a fixed time $t = 0$. Note that this state can be put in the form (1). Find $\psi(\mathbf{p})/\sqrt{2E_\mathbf{p}}$ for this state, and show that $\Psi(\mathbf{x}, t = 0)$ can be expressed in the form of a Fourier integral. Argue that the state $\phi(\mathbf{b})\,|0\rangle$ is not perfectly localized to a point at $\mathbf{x} = \mathbf{b}$.

e) Define a position wave function as in nonrelativistic quantum mechanics: $\psi(x) \equiv \langle x|\psi\rangle$, where $|x\rangle = \phi(x)\,|0\rangle$. Show that $\psi(x)$ is the same function as $\Psi(x)$ in (4).

Hint:

$$\psi(x) = \langle x|\psi\rangle = \langle 0|\,\phi(x) a^\dagger_\psi\,|0\rangle\,.$$

Substitute the expression for $\phi(x)$ and note that $[a_\mathbf{p}, a^\dagger_\psi] = \psi(\mathbf{p})$.

f) Prove that the wave function $\Psi(x)$ in (4) obeys the Schrödinger equation in the nonrelativistic limit $|\mathbf{p}| \ll m$. (More precisely, we require $|\mathbf{p}| \ll m$ for all $\mathbf{p}$'s contributing to the integral in (4).)

Hint: It is tempting to approximate $E_\mathbf{p} \approx m$ in this limit, but then we get a trivial time dependence e^{-imt}, which gives no time dependence for $|\Psi(x)|^2$. So we need to go to the next order:

$$E_\mathbf{p} = m\sqrt{1 + \mathbf{p}^2/m^2} \approx m + \mathbf{p}^2/2m$$

Then substitute in (4), ignore the trivial time dependence due to m, and verify that (4) satisfies the Schrödringer equation.

g) Consider a particle with narrow wavepacket spectrum about some central value $\mathbf{p}_0$. Let $\mathbf{p}_0$ point in the z-direction, and assume that the spectrum $\psi(\mathbf{p})$ only contains values for $\mathbf{p}$ along the z-direction. In other words, consider the 1+1d case

$$\Psi(z, t) = \int \frac{dp}{2\pi} \frac{\psi(p)}{\sqrt{2E}} e^{-iEt+ipz}, \quad E = \sqrt{p^2 + m^2}.$$

If $\psi(p)/\sqrt{2E}$ is a narrow gaussian spectrum centered about $p = p_0$, make a rough sketch of $\mathrm{Re}\,\Psi(z, t = 0)$. Indicate how the width of the pulse is related to the width of the gaussian spectrum.

h) Still in 1+1d, we now consider the propagation of the wavepacket in the ultrarelativistic limit $p \gg m$. Prove that in this case the pulse moves to the right (in the $+z$-direction), with no dispersion/distortion.

i) We now consider the propagation of $\Psi(z,t)$ in the nonrelativistic limit $p \ll m$. As before, we do not care what the overall constant is, we are only interested in the overall shape and width of $|\Psi(z,t)|^2$. Let $\psi(p)/\sqrt{2E}$ be a narrow gaussian, and use your favorite methods (by hand, or using an algebraic or numeric computer program) to obtain rough plots of $|\Psi(z,t)|^2$ as a function of z for a $t \ll m\sigma^2$ and a $t \gg m\sigma^2$. Or, since this is a little bit boring, perhaps just search the Internet for something like "gaussian wave packet in Schrödinger equation". For example there is a nice Wikipedia page "wave packet".

What happens with the pulse as the time goes by? What is the time $t \approx m\sigma^2$ in SI units?

j) Consider (4) in the 2+1d case

$$\Psi(y,z) = \int \frac{dp_y dp_z}{(2\pi)^2} \frac{\psi(\mathbf{p})}{\sqrt{2E_\mathbf{p}}} e^{-iE_\mathbf{p}t + ip_y y + ip_z z}. \tag{5}$$

Let $\psi(\mathbf{p})/\sqrt{2E_\mathbf{p}} = \delta(p_z - p_0) e^{-\frac{p_y^2}{2(\Delta p)^2}}$. How does the wave envelope $|\Psi(y,z)|^2$ look like for $t = 0$? What happens if we include a factor $e^{-ip_y b}$ in the expression for $\psi(\mathbf{p})/\sqrt{2E_\mathbf{p}}$?

k) In scattering experiments, which are quite central in QFT, two particles are initially far apart, but overlap after some time. We consider 2+1d or 3+1d. Why do you think we need to assume that the particles are contained in wavepackets? In other words, why can't we just assume that the particles each have a single momentum, p_1 and p_2?

l) Still, in scattering experiments the wavepackets have usually narrow momentum spectra such that they are much like plane waves. More precisely, $|\Delta\mathbf{p}| \ll |\mathbf{p}_0|$, where $\Delta\mathbf{p}$ is the variation in momentum. This means that the size of the pulse is much larger than a de Broglie wavelength, in all directions. Even though we cannot use true plane waves in the analysis, we can come as close as we wish to plane waves, by setting up the in states in sufficiently remote past, sufficiently far away.

In Peskin & Schroeder p. 102-104, the S-matrix is introduced as the overlap between so-called in and out states. Read p. 102-103 and the first paragraph on p. 104. Try to understand what is meant by the in and out states (and (4.70)). Consider the special case with a free Klein-Gordon theory (which is the one treated in this exercise). What is the S-matrix in this case? Relate to this exercise.

> **Hint:** When there is no interaction, the set of in states and out states are identical. Thus the S-matrix is the identity. When there is interaction, the in states are like free almost-plane-waves in the remote past, while the out states are like free almost-plane-waves in the remote future. However, they are Heisenberg states, with a common reference time for the Heisenberg picture.