University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 4

Problem 1 Group theory, part 1: Group conditions (T. Klungland)

In this exercise you are asked to show that $\mathrm{SU}(N)$, the group of unitary $N \times N$ with determinant equal to 1, satisfy the definition of a group. Condition 2 holds automatically like it did for the general linear group; you need to show the rest.

NB: This problem and the next are based on the group theory note posted on the course website; it is recommended you read it before solving.

a) Show that $\mathrm{SU}(N)$ satisfies condition 1, that is: For any $U_1, U_2 \in \mathrm{SU}(N)$, $U_3 \equiv U_1 U_2 \in \mathrm{SU}(N)$, meaning that $U_3^\dagger U_3 = I$ and $\det U_3 = 1$.

b) Show that $\mathrm{SU}(N)$ satisfies condition 3, i.e. that $I_{N\times N} \in \mathrm{SU}(N)$.

c) Show that $\mathrm{SU}(N)$ satisfies condition 4. For each $U \in \mathrm{SU}(N)$, an inverse exists by definition since $U^\dagger U = I$, but it must be verified that $U^{-1} = U^\dagger \in \mathrm{SU}(N)$.

Problem 2 Group theory, part 2: SU(2) algebra (T. Klungland)

This exercise explores the algebra of $\mathrm{SU}(2)$, and the generators of its fundamental and adjoint representations.

a) Consider an element of the fundamental representation of $\mathrm{SU}(2)$ expanded infinitesimally away from the identity:

$$U = I_{2\times 2} + i a_j u_j, \tag{1}$$

where $a_j \in \mathbb{R}$, $j = 1, 2, 3$, are infinitesimal parameters, and u_j are the generators of the fundamental representation. These provide a basis for the space of all such elements (argue why three basis elements are sufficient!). By requiring that $U \in \mathrm{SU}(2)$, keeping only the first order in a_j, show that the generators must satisfy the conditions

$$u_i^\dagger = u_i, \tag{2}$$

$$\mathrm{Tr}(u_i) = 0. \tag{3}$$

Hint: Note that the parameters a_j are arbitrary, meaning that $U = I_{2\times 2} + i a_j u_j$ must satisfy $\det U = 1$ and $U^\dagger U = I_{2\times 2}$ for any set of parameters.

b) One possible basis of the algebra is given by the Pauli matrices, $u_i = \frac{1}{2}\sigma_i$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Argue that these actually are a basis for the algebra, and show explicitly that it satisfies

$$[u_i, u_j] = i\epsilon_{ijk} u_k, \tag{5}$$

where ϵ_{ijk} is the Levi-Civita symbol.

c) For the interested student: Construct the generators of the adjoint representation according to the definition at the end of the group theory note, and show that these satisfy (5).

Problem 3 Poincare algebra (J. Skaar)

The Lorentz transformations form a group, the Lorentz group, consisting of rotations and boosts: $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$. More generally, one often considers the *Poincare group*, which also includes translations $x^\mu \mapsto x'^\mu = x^\mu + a^\mu$.

When we do a Lorentz transformation Λ, the quantum states undergo a unitary transformation $U(\Lambda)$ (P&S p. 23; will also be discussed on p. 59). Similarly, because spacetime is invariant under translations, translation by a four vector a leads to a unitary transformation on the quantum states.

We will use a notation $U(\Lambda, a)$ which denotes the quantum unitary transformation which results from a Lorentz transformation Λ followed by translation a:

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{6}$$

If you would like a detailed treatment, see Weinberg Ch. 2.

a) Argue that

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2). \tag{7}$$

Hint: The U matrices must form a representation of the Poincare group. In other words, the transformation $U(\Lambda_2, a_2)U(\Lambda_1, a_1)$ corrsponds to first doing a Lorentz transformation Λ_1, then translation a_1, then Lorentz transformation Λ_2, and finally translation a_2. Find the resulting Poincare transformation $x^\mu \mapsto x'^\mu$, and then the resulting U.

b) For an infinitesimal translation $a^\mu = \epsilon^\mu$, the unitary operator must be of the form

$$U(1, \epsilon) = 1 - i\epsilon_\mu P^\mu + \mathcal{O}(\epsilon^2), \tag{8}$$

where the terms $\mathcal{O}(\epsilon^2)$ can be dropped. Here P^μ is an operator independent of ϵ, and a factor $-i$ has been taken out for later convenience. We will later identify P^μ as the four-momentum operator.

Prove that P^μ is hermittian.

Hint: Use that $U(1, \epsilon)$ is unitary, and ignore terms second order in ϵ.

c) Prove that

$$U^{-1}(\Lambda,a)P^{\mu}U(\Lambda,a) = \Lambda^{\mu}{}_{\nu}P^{\nu}. \tag{9}$$

Eq. (9) means that the operator P^{μ} transforms under Lorentz transformations as a four-vector.

> **Hint:** Since the matrices U form a representation of the Poincare group, we must have
>
> $$U^{-1}(\Lambda,a)U(1,\epsilon)U(\Lambda,a) = U(1,\Lambda^{-1}\epsilon) \tag{10}$$
>
> First prove this relation; then use it.

d) Prove that

$$[P^{\mu},P^{\nu}] = 0 \tag{11}$$

for all μ and ν.

> **Hint:** Use (9), set $\Lambda = 1$ and $a = \epsilon$ (where ϵ is infinitesimal).

e) Similarly to the method we have used to prove (11), we can obtain the remaining Poincare algebra. Let $\Lambda = 1+\omega$, where ω is infinitesimal, and $a = \epsilon$. Then

$$U(1+\omega,\epsilon) = 1 - i\epsilon_{\mu}P^{\mu} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}, \tag{12}$$

for some infinitesimal coefficients $\frac{i}{2}\omega_{\mu\nu}$ and operators $J^{\mu\nu}$. Going through the same steps as above, one obtains (you don't have to verify these equations):

$$i[J^{\mu\nu},J^{\rho\sigma}] = g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\sigma\mu}J^{\rho\nu} + g^{\sigma\nu}J^{\rho\mu}, \tag{13a}$$

$$i[P^{\mu},J^{\rho\sigma}] = g^{\mu\rho}P^{\sigma} - g^{\mu\sigma}P^{\rho}, \tag{13b}$$

$$[P^{\mu},P^{\nu}] = 0. \tag{13c}$$

The set (13) is the so-called Lie algebra of the Poincare group.

Prove that $\omega_{\mu\nu} = -\omega_{\nu\mu}$, and therefore, that we can take $J^{\nu\mu} = -J^{\mu\nu}$.

> **Hint:** Use the condition for the Lorentz transformation matrix
>
> $$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}, \tag{14}$$
>
> as you learned in FYS3120. Also convince yourself that the antisymmetry $\omega_{\mu\nu} = -\omega_{\nu\mu}$ means that any symmetric part of $J^{\mu\nu}$ does not contribute to the product $\omega_{\mu\nu}J^{\mu\nu}$.

f) The Lie algebra describes the behavior of the group elements close to identity, (12). We will now consider more general group elements. Consider, for example, $U(1,a)$ for any a:

$$U(1,a) = [U(1,a/n)]^{n} = \left[1 - ia_{\mu}P^{\mu}/n + \mathcal{O}(n^{-2})\right]^{n}. \tag{15}$$

By letting $n\to\infty$, argue that

$$U(1,a) = e^{-ia_{\mu}P^{\mu}}. \tag{16}$$

What is the physical interpretation of P^{0}?

g) Prove that

$$e^{iPx}\phi(0)e^{-iPx} = \phi(x). \tag{17}$$

> **Hint:** Use the unitary translation operator (16).

h) Remarkably, the conclusions and interpretations so far have been found from the Poincare symmetry of the theory only; thus the results are valid even though we don't know the details of the theory. In particular they are valid for interacting theories.

Why can we choose the eigenstates of the Hamiltonian $H \equiv P^0$ to be simultaneous eigenstates of P^i?

> **Hint:** Eq. (11).

i) For the free-field ladder operators, use (17) to verify

$$e^{iPx}a_{\mathbf{p}}e^{-iPx} = a_{\mathbf{p}}e^{-ipx}, \tag{18a}$$

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} = a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t}, \tag{18b}$$

$$e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{18c}$$

which correspond to (2.46) and (2.48) in P&S.