University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 4

Problem 1 Group theory, part 1: Group conditions (T. Klungland)

In this exercise you are asked to show that $\mathrm{SU}(N)$, the group of unitary $N \times N$ with determinant equal to 1, satisfy the definition of a group. Condition 2 holds automatically like it did for the general linear group; you need to show the rest.

NB: This problem and the next are based on the group theory note posted on the course website; it is recommended you read it before solving.

a) Show that $\mathrm{SU}(N)$ satisfies condition 1, that is: For any $U_1, U_2 \in \mathrm{SU}(N)$, $U_3 \equiv U_1 U_2 \in \mathrm{SU}(N)$, meaning that $U_3^\dagger U_3 = I$ and $\det U_3 = 1$.

Solution: The hermitian conjugate of a product is $(AB)^\dagger = B^\dagger A^\dagger$, so

$$U_3^\dagger U_3 = U_2^\dagger U_1^\dagger U_1 U_2 = I, \tag{1}$$

proving the "unitary" part of $\mathrm{SU}(N)$. Second, since $\det(AB) = \det A \det B$,

$$\det U_3 = \det U_1 \det U_2, \tag{2}$$

completing the proof.

b) Show that $\mathrm{SU}(N)$ satisfies condition 3, i.e. that $I_{N\times N} \in \mathrm{SU}(N)$.

Solution: Firstly, $\det I = 1$. Secondly, $I^\dagger = I$, meaning that

$$I^\dagger I = I^2 = I, \tag{3}$$

so that $I \in \mathrm{SU}(N)$.

c) Show that $\mathrm{SU}(N)$ satisfies condition 4. For each $U \in \mathrm{SU}(N)$, an inverse exists by definition since $U^\dagger U = I$, but it must be verified that $U^{-1} = U^\dagger \in \mathrm{SU}(N)$.

Solution: As stated in the exercise, the inverse of U automatically exists and is given by $U^{-1} = U^\dagger$. With this, we have

$$\left(U^{-1}\right)^\dagger U^{-1} = \left(U^\dagger\right)^\dagger U^\dagger = U U^\dagger = I, \tag{4}$$

and

$$\det\left(U^{-1}\right) = (\det U)^{-1} = 1. \tag{5}$$

Problem 2 Group theory, part 2: SU(2) algebra

(T. Klungland)

This exercise explores the algebra of SU (2), and the generators of its fundamental and adjoint representations.

a) Consider an element of the fundamental representation of SU (2) expanded infinitesimally away from the identity:

$$U = I_{2\times 2} + ia_j u_j, \tag{6}$$

where $a_j \in \mathbb{R}$, $j = 1, 2, 3$, are infinitesimal parameters, and u_j are the generators of the fundamental representation. These provide a basis for the space of all such elements (argue why three basis elements are sufficient!). By requiring that $U \in \mathrm{SU}(2)$, keeping only the first order in a_j, show that the generators must satisfy the conditions

$$u_i^\dagger = u_i, \tag{7}$$
$$\mathrm{Tr}\,(u_i) = 0. \tag{8}$$

Hint: Note that the parameters a_j are arbitrary, meaning that $U = I_{2\times 2} + ia_j u_j$ must satisfy $\det U = 1$ and $U^\dagger U = I_{2\times 2}$ for any set of parameters.

Solution: We begin by arguing why three numbers are required to parametrize all matrices in SU (2). A general 2×2 matrix has eight degrees of freedom, since all four elements can be complex. We then require unitarity, which means enforcing $U^\dagger U = I_{2\times 2}$. This gives four equations to restrict the parameters, lowering the number of degrees of freedom by four. Requiring the determinant to be 1 gives an additional constraint, bringing the number of degrees of freedom, and hence the number of parameters required to describe any SU (2) matrix, down to three.

Moving to the main part of the exercise, we first require unitarity:

$$1 = U^\dagger U = \left(I - ia_j u_j^\dagger\right)\left(I + ia_k u_k\right) \tag{9}$$
$$= I + ia_j\left(u_j - u_j^\dagger\right) + \mathcal{O}\left(a^2\right), \tag{10}$$

meaning that we must have $u_j = u_j^\dagger$ for $j = 1, 2, 3$ for this to hold for any combination of parameters a_j.

To write the determinant of U, we parametrize U as

$$U = \begin{pmatrix} 1 + ia_j A_j & ia_j B_j \\ ia_j C_j & 1 + ia_j D_j \end{pmatrix}, \tag{11}$$

where we parametrized the generators by $u_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$. The determinant of U is then

$$\det U = 1 + ia_j \underbrace{\left(A_j + D_j\right)}_{=\mathrm{Tr}(u_j)} + \mathcal{O}\left(a^2\right). \tag{12}$$

Requiring this to be 1 independent of the parameters a_i then leads to the condition $\mathrm{Tr}\,(u_i) = 0$.

b) One possible basis of the algebra is given by the Pauli matrices, $u_i = \frac{1}{2}\sigma_i$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

Argue that these actually are a basis for the algebra, and show explicitly that it satisfies

$$[u_i, u_j] = i\epsilon_{ijk}u_k, \tag{14}$$

where ϵ_{ijk} is the Levi-Civita symbol.

Solution: First off, the Pauli matrices are an appropriate basis since they satisfy the conditions derived in part a, and because they are linearly independent.

Straightforward matrix multiplication reveals that

$$[u_1, u_2] = -[u_2, u_1] = iu_3, \tag{15}$$
$$[u_2, u_3] = -[u_3, u_2] = iu_1, \tag{16}$$
$$[u_3, u_1] = -[u_1, u_3] = iu_2, \tag{17}$$

which can be summarized as $[u_i, u_j] = i\epsilon_{ijk}u_k$.

c) For the interested student: Construct the generators of the adjoint representation according to the definition at the end of the group theory note, and show that these satisfy (14).

Solution: The definition of the adjoint representation gives $\left(u_i^A\right)_{jk} = -i\epsilon_{ijk}$; explicitly, this gives

$$u_1^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{18}$$

$$u_2^A = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{19}$$

$$u_3^A = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

Showing that these satisfy $\left[u_i^A, u_j^A\right] = i\epsilon_{ijk}u_k^A$ is again a matter of straightforward matrix multiplication.

Problem 3 Poincare algebra (J. Skaar)

The Lorentz transformations form a group, the Lorentz group, consisting of rotations and boosts: $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$. More generally, one often considers the *Poincare group*, which also includes translations $x^\mu \mapsto x'^\mu = x^\mu + a^\mu$.

When we do a Lorentz transformation Λ, the quantum states undergo a unitary transformation $U(\Lambda)$ (P&S p. 23; will also be discussed on p. 59). Similarly, because spacetime is invariant under translations, translation by a four vector a leads to a unitary transformation on the quantum states.

We will use a notation $U(\Lambda, a)$ which denotes the quantum unitary transformation which results from a Lorentz transformation Λ followed by translation a:

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{21}$$

If you would like a detailed treatment, see Weinberg Ch. 2.

a) Argue that

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2). \tag{22}$$

Hint: The U matrices must form a representation of the Poincare group. In other words, the transformation $U(\Lambda_2, a_2)U(\Lambda_1, a_1)$ corrsponds to first doing a Lorentz transformation Λ_1, then translation a_1, then Lorentz transformation Λ_2, and finally translation a_2. Find the resulting Poincare transformation $x^\mu \mapsto x'^\mu$, and then the resulting U.

b) For an infinitesimal translation $a^\mu = \epsilon^\mu$, the unitary operator must be of the form

$$U(1, \epsilon) = 1 - i\epsilon_\mu P^\mu + \mathcal{O}(\epsilon^2), \tag{23}$$

where the terms $\mathcal{O}(\epsilon^2)$ can be dropped. Here P^μ is an operator independent of ϵ, and a factor $-i$ has been taken out for later convenience. We will later identify P^μ as the four-momentum operator.

Prove that P^μ is hermittian.

Hint: Use that $U(1, \epsilon)$ is unitary, and ignore terms second order in ϵ.

c) Prove that

$$U^{-1}(\Lambda, a)P^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu P^\nu. \tag{24}$$

Eq. (24) means that the operator P^μ transforms under Lorentz transformations as a four-vector.

Hint: Since the matrices U form a representation of the Poincare group, we must have

$$U^{-1}(\Lambda, a)U(1, \epsilon)U(\Lambda, a) = U(1, \Lambda^{-1}\epsilon) \tag{25}$$

First prove this relation; then use it.

Solution: The composed operator on the left-hand side of (25) corresponds to the total Lorentz transformation

$$x \mapsto x' = \Lambda^{-1}\big([(\Lambda x + a) + \epsilon] - a\big) = x + \Lambda^{-1}\epsilon.$$

(Remember that corresponding to $U^{-1}(\Lambda, a)$ we must first subtract a before applying Λ^{-1}.) Thus we obtain (25). Now substitute (23) in (25) and note that ϵ is arbitrary.

d) Prove that

$$[P^\mu, P^\nu] = 0 \tag{26}$$

for all μ and ν.

Hint: Use (24), set $\Lambda = 1$ and $a = \epsilon$ (where ϵ is infinitesimal).

e) Similarly to the method we have used to prove (26), we can obtain the remaining Poincare algebra. Let $\Lambda = 1 + \omega$, where ω is infinitesimal, and $a = \epsilon$. Then

$$U(1 + \omega, \epsilon) = 1 - i\epsilon_\mu P^\mu + \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}, \tag{27}$$

for some infinitesimal coefficients $\frac{i}{2}\omega_{\mu\nu}$ and operators $J^{\mu\nu}$. Going through the same steps as above, one obtains (you don't have to verify these equations):

$$i[J^{\mu\nu}, J^{\rho\sigma}] = g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\sigma\mu} J^{\rho\nu} + g^{\sigma\nu} J^{\rho\mu}, \tag{28a}$$
$$i[P^\mu, J^{\rho\sigma}] = g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho, \tag{28b}$$
$$[P^\mu, P^\nu] = 0. \tag{28c}$$

The set (28) is the so-called Lie algebra of the Poincare group.

Prove that $\omega_{\mu\nu} = -\omega_{\nu\mu}$, and therefore, that we can take $J^{\nu\mu} = -J^{\mu\nu}$.

Hint: Use the condition for the Lorentz transformation matrix

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}, \tag{29}$$

as you learned in FYS3120. Also convince yourself that the antisymmetry $\omega_{\mu\nu} = -\omega_{\nu\mu}$ means that any symmetric part of $J^{\mu\nu}$ does not contribute to the product $\omega_{\mu\nu} J^{\mu\nu}$.

Solution: For the first part use the hint:

$$g_{\rho\sigma} = g_{\mu\nu}\big(\delta^\mu{}_\rho + \omega^\mu{}_\rho\big)\left(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma\right) = g_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2). \tag{30}$$

Since ω is infinitesimal, we ignore the second order term and obtain the antisymmetry: $\omega_{\sigma\rho} + \omega_{\rho\sigma} = 0$.

For the second part, we decompose $J^{\mu\nu}$ into its symmetric and antisymmetric parts:

$$J^{\mu\nu} = \frac{1}{2}(J^{\mu\nu} + J^{\nu\mu}) + \frac{1}{2}(J^{\mu\nu} - J^{\nu\mu}) . \tag{31}$$

Then we note that the symmetric parts gives zero after multiplication by $\omega_{\mu\nu}$:

$$\begin{aligned} \omega_{\mu\nu}(J^{\mu\nu} + J^{\nu\mu}) &= \omega_{\mu\nu} J^{\mu\nu} + \omega_{\mu\nu} J^{\nu\mu} = \omega_{\mu\nu} J^{\mu\nu} - \omega_{\nu\mu} J^{\nu\mu} \\ &= \omega_{\mu\nu} J^{\mu\nu} - \omega_{\mu\nu} J^{\mu\nu} = 0, \end{aligned} \tag{32}$$

where we have used the antisymmetry of ω in the second equality, and renamed the indices ($\mu \mapsto \nu$ and $\nu \mapsto \mu$) in the third. (Any index that is summed over can be named whatever you like.) Thus the symmetric part of $J^{\mu\nu}$ can be taken to be zero in (27).

f) The Lie algebra describes the behavior of the group elements close to identity, (27). We will now consider more general group elements. Consider, for example, $U(1, a)$ for any a:

$$U(1, a) = [U(1, a/n)]^n = \left[1 - ia_\mu P^\mu/n + \mathcal{O}(n^{-2})\right]^n . \tag{33}$$

By letting $n \to \infty$, argue that

$$U(1, a) = e^{-ia_\mu P^\mu} . \tag{34}$$

What is the physical interpretation of P^0?

Solution: We have

$$U(1, a) = [U(1, a/n)]^n = \left[1 - \frac{1}{n}\left(ia_\mu P^\mu + \mathcal{O}(n^{-1})\right)\right]^n , \tag{35}$$

which tends to (34) by the usual property of the exponential function. Letting $a^\mu = (t, 0, 0, 0)$ we have $U(1, a) = e^{-iP^0 t}$, which acts on the quantum state. Thus $P^0 = H$ is the Hamiltonian. It is then natural to identify P^μ as the four-momentum vector.

g) Prove that

$$e^{iPx}\phi(0)e^{-iPx} = \phi(x). \tag{36}$$

Hint: Use the unitary translation operator (34).

Solution: Considering, for example, the expectation value $\langle\psi|\,\phi(x)\,|\psi\rangle$ in an arbitrary state $|\psi\rangle$, at the point $x = 0$. Due to translation invariance of spacetime, we may move the entire experiment from 0 to a, without changing the expectation value. Thus we want $\phi(x) \mapsto \phi(x-a)$, which in particular means that $\phi(a) \mapsto \phi(0)$. (Perhaps you don't immediately agree here; transformations are indeed confusing. Try to convince yourself by plotting a function $f(x)$ with a peak at $x = 0$; then $f(x - a)$ will have the peak for $x = a$.)

We translate $x^\mu \mapsto x'^\mu = x^\mu + a^\mu$, which means that the quantum state transforms as

$$|\psi\rangle \mapsto U(1, a^\mu) |\psi\rangle = e^{-iP^\mu a_\mu} |\psi\rangle = e^{-iPa} |\psi\rangle . \tag{37}$$

Thus the expectation value transforms as

$$\langle\psi| \phi(a) |\psi\rangle \mapsto \langle\psi| e^{iPa}\phi(0)e^{-iPa} |\psi\rangle \tag{38}$$

Since the expectation value must be unchanged, we have

$$\langle\psi| \phi(a) |\psi\rangle = \langle\psi| e^{iPa}\phi(0)e^{-iPa} |\psi\rangle . \tag{39}$$

Since this must be valid for any $|\psi\rangle$ (and since $\phi(x)$ is hermitian and therefore diagonalizable), we must have $\phi(a) = e^{iPa}\phi(0)e^{-iPa}$. The translation vector a is arbitrary, so we may instead write

$$\phi(x) = e^{iPx}\phi(0)e^{-iPx}. \tag{40}$$

h) Remarkably, the conclusions and interpretations so far have been found from the Poincare symmetry of the theory only; thus the results are valid even though we don't know the details of the theory. In particular they are valid for interacting theories.

Why can we choose the eigenstates of the Hamiltonian $H \equiv P^0$ to be simultaneous eigenstates of P^i?

Hint: Eq. (26).

i) For the free-field ladder operators, use (36) to verify

$$e^{iPx}a_{\mathbf{p}}e^{-iPx} = a_{\mathbf{p}}e^{-ipx}, \tag{41a}$$

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} = a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t}, \tag{41b}$$

$$e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{41c}$$

which correspond to (2.46) and (2.48) in P&S.

Solution: Express the ladder operator from the field (see problem set 2), and use (36). The last two relations are special cases.