University of Oslo

FYS4170/9170 — Relativistic Quantum Field Theory

Problem set 5

Problem 1 Classical field theory for the Dirac field

(J. Skaar)

The Lagrangian for the Dirac field is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{1}$$

Here $\psi = \psi(x)$ is a wave function object consisting of 4 elements, $\bar{\psi} = \psi^\dagger \gamma_0$, and γ^μ for $\mu = 0, 1, 2, 3$ are constant 4×4 matrices. Note that while μ is a spacetime index, the four elements ψ_α of ψ do *not* correspond to the four dimensions of spacetime.

a) Argue that (1) can be written in the form

$$\mathcal{L} = \bar{\psi}_\alpha i\gamma^\mu_{\alpha\beta} \partial_\mu \psi_\beta - m\bar{\psi}_\alpha \psi_\alpha, \tag{2}$$

where we have used the Einstein summation convention, and the indices α and β run over the four elements of the vectors.

b) Find the Euler-Lagrange equation for ψ and $\bar{\psi}$ by treating ψ and $\bar{\psi}$ (or $\psi^\dagger$) as independent.

Solution: The Euler-Lagrange equation for a field ϕ is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

We get one equation for each field ψ_β:

$$\partial_\mu \bar{\psi}_\alpha i\gamma^\mu_{\alpha\beta} + m\bar{\psi}_\beta = 0, \tag{3}$$

and one for each field $\bar{\psi}_\alpha$:

$$i\gamma^\mu_{\alpha\beta} \partial_\mu \psi_\beta - m\psi_\alpha = 0. \tag{4}$$

The latter equation can be written

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0, \tag{5}$$

which is the Dirac equation, while the former can be shown by properties of the γ matrices ($(\gamma^0)^2 = 1$, $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$) to be Dirac equation in Hermitian-conjugate form.

c) The Lagrangian is unchanged under a global phase transformation

$$\psi(x) \to e^{-i\alpha}\psi(x), \tag{6}$$

where the constant α is real. What is the associated, conserved Noether current density?

Solution: Referring to p. 17-18 in P&S, we have

$$\alpha\Delta\psi_\beta = -i\alpha\psi_\beta$$
$$\alpha\Delta\bar{\psi}_\beta = i\alpha\bar{\psi}_\beta$$
$$\mathcal{J}^\mu = 0.$$

The conserved current is (see (2.12) and the comment below it, in P&S)

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\beta)}(-i\psi_\beta) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_\beta)}i\bar{\psi}_\beta = -\bar{\psi}_\alpha i\gamma^\mu_{\alpha\beta}i\psi_\beta = \bar{\psi}\gamma^\mu\psi. \tag{7}$$

Problem 2 Dirac matrices (L. L. Braseth, T. Klungland)

This problem is intended to give you some practice working with Dirac matrices, as well as to show the origin of some identities that will be very useful in future calculations.

All of the identities can be found from the Dirac algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4\times4}. \tag{8}$$

Note that the identity matrix on the right-hand side is usually left implicit, but in certain situations (for example in part b) it is important to remember that it is there.

a) Show the following contraction identities:

$$\gamma^\mu\gamma_\mu = 4, \tag{9}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \tag{10}$$
$$\gamma^\mu\gamma^\nu\gamma^\alpha\gamma_\mu = 4g^{\nu\alpha}, \tag{11}$$
$$\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho\gamma_\mu = -2\gamma^\rho\gamma^\alpha\gamma^\nu. \tag{12}$$

Hint: Begin by showing (9); then, for each successive identity, use the previous one to shorten the calculations.

Solution: Eq. (9) follows from (8) since

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = g_{\mu\nu}g^{\mu\nu} = 4. \tag{13}$$

As an aside, the second equality is a consequence of symmetry: Since $g_{\mu\nu}$ is symmetric in its indices, only the symmetric part of $\gamma^\mu\gamma^\nu$ can contribute in the contraction: Using some convenient notation that is frequently used in such calculations, we have that for any symmetric $A_{\mu\nu}$ and arbitrary $B^{\mu\nu}$,

$$A_{\mu\nu}B^{\mu\nu} = A_{\mu\nu}B^{(\mu\nu)}, \tag{14}$$

where $B^{(\mu\nu)}$ is the symmetric component of $B^{\mu\nu}$, defined as $B^{(\mu\nu)} \equiv \frac{1}{2}\left(B^{\mu\nu} + B^{\nu\mu}\right)$. Its antisymmetric component is similarly defined as $B^{[\mu\nu]} \equiv \frac{1}{2}\left(B^{\mu\nu} - B^{\nu\mu}\right)$.

With the first identity at hand, we obtain (10) from the Dirac algebra by writing $\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$; the first term then becomes trivial, while the second can be solved using (9). The remaining identities follow in the same way — (11) follows from (10), and (12) follows from (11) — with some additional use of (8) to re-arrange matrices.

b) Calculations involving fermions typically involve evaluating traces of Dirac matrices, and products of such. These calculations can be sped up substantially by recalling some properties of these traces; to that end, derive the following relations by using the Dirac algebra and the cyclic property of traces, $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(CAB)$. For the last identity, the γ^5 matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, which satisfies $\left(\gamma^5\right)^2 = \mathbb{1}_{4\times4}$, $\left\{\gamma^5, \gamma^\mu\right\} = 0$, may be useful.

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = 4g^{\mu\nu}, \tag{15}$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}, \tag{16}$$

$$\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right] = 0, \text{ for } n = 2m-1 \text{ (odd)}. \tag{17}$$

Hint: The second identity follows from the first; to show the last identity, insert a factor of $\left(\gamma^5\right)^2$ and then use the cyclic property of the trace, and the anticommutation of γ^5 with the other matrices.

Solution: The first identity is shown from the Dirac algebra and the cyclic property of traces:

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = \frac{1}{2}\mathrm{Tr}\left[\{\gamma^\mu, \gamma^\nu\}\right] = g^{\mu\nu}\mathrm{Tr}\left[\mathbb{1}_{4\times4}\right] = 4g^{\mu\nu}. \tag{18}$$

For the next one, commute γ^μ past each of the others until it is furthest to the right, each time using (8); this leaves three terms where you can use the equation we just derived, and one which, once you cycle γ^μ back to the front, is identical to the original trace.

To show that the trace of any odd-numbered product of γ matrices vanishes, insert a factor $\mathbb{1} = \gamma^5\gamma^5$ to the right of the $n = 2m-1$ other matrices, and cycle one γ^5 to the front, which doesn't change the result. Then move the leftmost γ^5 past all of the other matrices; since it anticommutes with all of them, this gives an additional factor $(-1)^{2m-1} = -1$. You are then left with the same trace that you started with (removing again the factor $\gamma^5\gamma^5$), only with a minus sign; thus it must be zero.

c) Show the following identities involving contractions between Dirac matrices and four-momenta, $p_\mu\gamma^\mu \equiv \not{p}$:

$$(\not{p})^2 = p^2, \tag{19}$$

$$\mathrm{Tr}\left[\not{p}_1\not{p}_2\not{p}_3\not{p}_4\right] = 4\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right) - 4\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right) + 4\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right). \tag{20}$$

Hint: Use the symmetry considerations discussed in the solution to part a, and results from the previous parts.

Solution: First, using the same symmetry arguments as before, we have

$$(\not{p})^2 = p_\mu \gamma^\mu p_\nu \gamma^\nu = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = p_\mu p_\nu g^{\mu\nu} = p^2, \tag{21}$$

since $p_\mu p_\nu = p_\nu p_\mu$ (recall that these are components of the four-momentum, i.e. simply numbers). The trace identity follows from (16) by writing $\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4 = p_{1\mu} p_{2\nu} p_{3\rho} p_{4\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$ and using the linearity of traces to factor out the momenta.

Problem 3 Classical source of particles (J. Skaar)

In this problem we will consider the creation of Klein-Gordon particles with a classical current source $j(x)$, see P&S p. 32. The creation of photons with a classical, electric current source will be similar.

We assume that the current source has acted for some time, but that it has been turned off before the observation time x^0. You should start by going through (and writing out) the derivation of (2.64) in P&S p. 32.

Recall that $\tilde{j}(p)$ is the Fourier transform of $j(x)$, and that the Heisenberg-picture field transforms from its usual free form

$$\phi_{\text{free}}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left\{ a_{\mathbf{p}} e^{-ipx} + \text{h.c.} \right\} \tag{22}$$

to

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left\{ [a_{\mathbf{p}} + \alpha \psi(\mathbf{p})] e^{-ipx} + \text{h.c.} \right\} \tag{23}$$

by the action of the source. Here

$$\alpha \psi(\mathbf{p}) = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \tilde{j}(p), \tag{24}$$

where $p^2 = m^2$. The parameter α is included such that $\psi(\mathbf{p})$ can be assumed normalized: $\int \frac{d^3 p}{(2\pi)^3} |\psi(\mathbf{p})|^2 = 1$.

You may find the following operator identiy useful:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \tag{25}$$

In addition you may need the Baker–Campbell–Hausdorff formula:

$$e^A e^B = e^C, \text{ where } C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B] - \frac{1}{12}[B, [A, B] + \ldots \tag{26}$$

a) Define the wavepacket ladder operator

$$a_\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \psi(\mathbf{p}) a_\mathbf{p}^\dagger, \tag{27}$$

and a so-called displacement operator

$$D(\alpha) = e^{\alpha a_\psi^\dagger - \alpha^* a_\psi} \tag{28}$$

Note that $D(\alpha)$ is unitary, and show that

$$D^\dagger(\alpha) a_\mathbf{p} D(\alpha) = a_\mathbf{p} + \alpha\psi(\mathbf{p}). \tag{29}$$

Hint: Calculate the commutator $[\alpha^* a_\psi - \alpha a_\psi^\dagger, a_\mathbf{p}]$ and use (25).

b) Eq. (29) implies that

$$\phi(x) = D^\dagger(\alpha)\phi_{\text{free}}(x)D(\alpha). \tag{30}$$

Thus the unitary source operator (or time-evolution operator) in the Heisenberg picture is $D(\alpha)$.

Going to the Schrödinger picture, describe the action of the source if we start in a vacuum state. Prove that the resulting state is a coherent state,

$$|\alpha\rangle_\psi = e^{-|\alpha|^2/2} \sum_n \frac{(\alpha a_\psi^\dagger)^n}{n!} |0\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle_\psi . \tag{31}$$

Here $|n\rangle_\psi$ denotes a n-photon state in the wavepacket ψ. This result is perhaps not very surprising; a classical source produces a classical (coherent) state. For photons this is an expected result: The electromagnetic state produced by a current (an antenna) is a coherent state, which describes a classical electromagnetic field.

Solution: The source produces the state

$$|\alpha\rangle_\psi \equiv D(\alpha)|0\rangle . \tag{32}$$

In order to evaluate this state, it is useful to rewrite $D(\alpha)$ such that the annihilation operators act first. This can be done with the help of the Baker–Campbell–Hausdorff formula, which becomes simple since the commutators of commutators vanish. Setting $A = \alpha a_\psi^\dagger$ and $B = -\alpha^* a_\psi$, this leads to

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a_\psi^\dagger} e^{-\alpha^* a_\psi} \tag{33}$$

With (33), the desired form (31) follows immediately since $a_\psi |0\rangle = 0$.