FYS4620

Obligatory exercise

10.10.2017

PRACTICAL INFORMATION

This set of problems is the obligatory part of the course *FYS4620 Introduction to plasma physics* lectured at the University of Oslo in H2017 semester. It contributes 15% to the final grade.
It is strongly encouraged that you work on the problems with your fellow students or even as the whole group. However, each student has to write and submit his/her own report.

- You have 3 weeks for solving this set of problems.
- You have to submit your report is by sending it to the following email: w.j.miloch@fys.uio.no (it can also be a scan).
- If you prefer, you can also write by hand and draw plots and figures by hand.
- You are encouraged to use literature (sometimes you will neet to find typical parameters for plasmas of interests), but remember to provide references to the sources.
- **The deadline for submission is 6 November at 23:59 hours.**

PROBLEM 1

Consider a cylindrical coordinate system, (r, y), where the y-axis is the symmetry axis. At the positions $y = -a$ and $y = a$ we have two parallel metal plates. The system is embedded in an inhomogeneous magnetic field, where we can assume, as an approximation, that all magnetic field lines emerge from a point $(r, y) = (0, -A)$, where $A \gg a$, see also Fig. 1. Note that by this assumption, we are *not* implying that there is spherical symmetry: the intensity

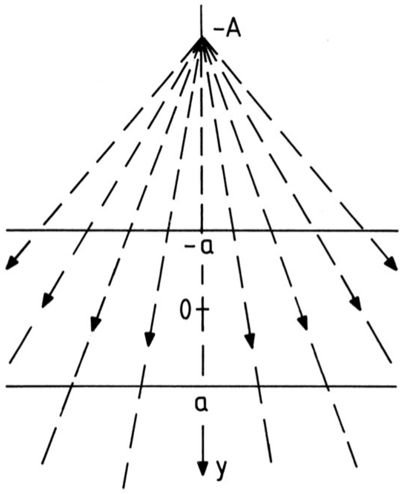

Figure 1: Illustration of the direction of the model magnetic field, and positions of conducting plates. The directions of the magnetic field lines are shown by dashed lines and arrows. The y-axis for the cylindrical symmetry is here vertical, and the r-axis horizontal.

of the magnetic field lines vary with direction, as explained in the following. In the plane perpendicular to the y-axis at the position $y = 0$ we assume to have the y-component of the magnetic field B_0 independent of the radial position r.

(a) Make a sketch of the magnetic field vectors along the line $y = 0$. Why is the proposed magnetic field only possible as an approximation? Hint: we have generally $\nabla \cdot \mathbf{B} = 0$.

(b) Note that we do not assume the system to be in a vacuum: there is a spatially distributed current system which maintains the magnetic field. Determine the spatially distributed current system, which gives rise to the postulated magnetic field.

(c) Demonstrate that with the given assumptions, see in particular also Fig. 1, we have the magnetic field component B_y independent of r everywhere.

(d) Determine B_y, $\partial B_y/\partial y$, and the radial magnetic field component B_r as a function of r as well as y between the plates. Give the exact expression, and present also a series expansion, accurate to first order in y/A.

A charged particle with mass M, charge q and velocity $\mathbf{U_0} \perp \hat{\mathbf{y}}$ is gyrating around the magnetic field lines at the origin, $(r, y) = (0, 0)$, with its gyro-center moving along the y-axis.

(e) Determine the particle gyro radius r_L and its magnetic moment.

(f) Because of the magnetic field inhomogeneity, the particle is subject to a force. Determine this force in magnitude as well as direction, assuming r_L to be small. Express the force in terms of $\partial B_y/\partial y$.

This force is now neutralized at $y = 0$ by charging the metal plates with positive and negative surface charges, $\pm\sigma$, respectively.

(g) How strong do you need the electric field to be, and what is the surface charge σ needed to achieve this field?

In a short time interval, the particle is now given a small velocity component $U_{||} \ll U_0$ along the magnetic field.

(h) How is the motion of the particle's gyrocenter, if we can assume that $\partial B_y/\partial y = \text{const}$?

(i) Describe the motion of the gyrocenter, when you for $\partial B_y/\partial y$ use the series expansion from question (d). It is assumed that $U_{||}$ is so small that the particle does not reach any of the conducting plates within time scales of interest.

PROBLEM 2

Write the expression for the critical pitch angle separating confined and free particles for a magnetic mirror in terms of the minimum and maximum magnetic fields between the two mirrors.
Approximate the Earth's magnetic field by a simple dipole field. To quantify the confinement of charged particles by the radiation belts of the Earth, plot the variation of the critical pitch angle for varying altitude at the Earth's magnetic equator starting at 3 Earth radii, assuming that in all cases the relevant maximum magnetic field is the one obtained over the magnetic pole at one Earth radius. Make a plot of the result.

PROBLEM 3

Consider the following partial differential equations and give for each of them the corresponding dispersion relation $\omega = \omega(k)$.

1. $\alpha^2 \frac{d^2}{dt^2}\Psi + B^2\Psi = \beta^2 \frac{d^2}{dx^2}\Psi$
2. $\alpha^2 \frac{d^2}{dt^2}\Psi + B^2\Psi = \beta^2 \frac{d^3}{dx^3}\Psi$
3. $\alpha^2 \frac{d^2}{dt^2}\Psi + B^2\Psi = \beta^2 \frac{d^4}{dx^4}\Psi$
4. $\alpha \frac{d}{dt}\Psi - C\frac{d}{dx}\Psi = \beta \frac{d^3}{dx^3}\Psi$
5. $\alpha \frac{d}{dt}\Psi - C\frac{d}{dx}\Psi^2 = \beta \frac{d^3}{dx^3}\Psi$

Describe the procedure and assumption in obtaining the dispersion relation. Make a sketch of the dispersion relations on $\omega - k$ diagrams. Be careful with the signs of α and β. What can you say about the phase velocity and group velocity for each case?
The examples above are written for one spatial dimension. Try to write cases 1,2, 5 and 6 for a fully three dimensional case using the ∇-operator.

PROBLEM 4

(a) Write the general MHD-expression for the space-time varying magnetic field for a given plasma velocity field, assuming finite conductivity. Which term can be ignored in ideal MHD?

(b) An observer finds that in a large volume of space the magnetic field has a constant direction (the z-direction) and varies linearly with time as $\mathbf{B} = \mathbf{B_0}(1 + t/\tau)$ for $t > 0$, where τ is constant. Assuming ideal MHD, give a velocity field that is consistent with this observation. Is the velocity field uniquely determined?

(c) What is the requirement for a velocity field being incompressible? Is this criterion fulfilled for the present problem?

PROBLEM 5

A spacecraft in plasma will be charged by electrons and ions and will be shielded by plasma particles where the characteristic shielding length can be related to the Debye length λ_D. Assume that the spacecraft size is much larger than the Debye length. Then we can use the so-called *thin-sheath* approximation. In the limiting case we can neglect the edge effects, and can consider a spacecraft as an infinite plate. The electrons are much more mobile, and the spacecraft will initially acquire negative charge and a negative potential with respect to plasma.

(a) Write general expressions for electron and ion currents to the surface.

(b) Find the electron current to the surface for electrons, assuming that they are Boltzmann distributed.

(c) Find the ion current to the surface for ions with the mean velocity determined by the thermal velocity of ions $v_{th} = \sqrt{\frac{kT_i}{m_i}}$.

(d) When the spacecraft is negative, less electrons will reach the surface, and at some potential, which is called floating potential, the ion and electron currents will be balanced Determine floatnig potential of the spacecraft for stationary conditions, i.e., when the net current to the surface is zero. Discuss what this potential depends on.

Let us now take also into consideration charing due to photoemission. Use literature to find the expression for the photoelectric current, and determine the photoelectric current to the surface. How does this modify the potential of the spacecraft? (again by considering

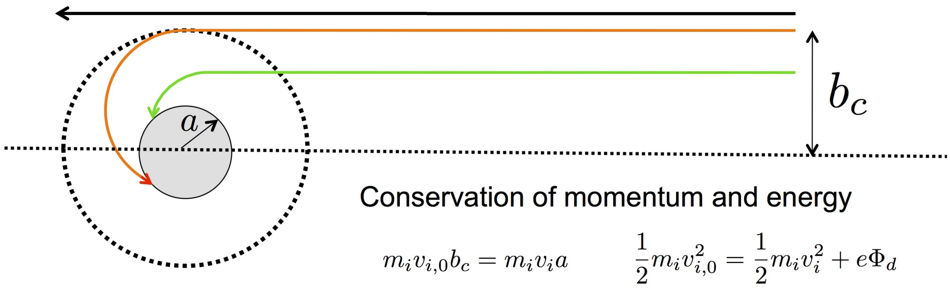

Figure 2: Schematics of ion trajectories in the model. b_c is the impact parameter, and a is the radius of the object and the poetnial of the object is Φ_d.

the vanishing net current to the surface). Take some typical parameters for the Low-Earth-Orbit, and provide an estimate of the spacecraft potential in such an orbit. Discuss your results. Discuss also how a charged spacecraft will affect the plasma in its vicinity.
Finally, let us take another limit and assume that the spacecraft is much smaller than the Debye length. For simplicity we can now approximate it by a sphere. We can further assume that every ion that enters the spherical sheath around the spacecraft will be neutralised at the surface and contribute to the current, see Fig. 2. Ions with initial velocity $v_{i,0}$ will have conserved energy and angular momentum, as shown in the figure.

(a) Calculate the impact parameter for the incoming ions b_c and find the collection cross section $\sigma_c = \pi b_c^2$.

(b) Find the infinitesimal ion current to the surface dI_i.

(c) Assume Maxwellian distribution for ions $f(v_i) = 4\pi v_i^2 \left(\frac{m_i}{2\pi k T_i}\right)^{3/2} \exp\left(-\frac{m_i v_i^2}{2kT_i}\right)$, with $\int_0^\infty f(v_i)\,dv_i = 1$, and find the total ion current to the surface. Note that the integral should be from 0 to ∞ and that $\int_0^\infty x^n e^{-ax^2} dx = \frac{k!}{2a^{k+1}}$ for odd $n = 2k+1$.

(d) Assume Boltzmann-distributed electrons and show that the expression for floating potential in this case can be given with: $1 - \frac{e\Phi_f}{kT_i} = \sqrt{\frac{m_i T_e}{m_e T_i}} \frac{n_e}{n_i} \exp\left(\frac{e\Phi_f}{kT_e}\right)$. Show that in the limiting case of isothermal hydrogen plasma, floating potential will reach the Spitzer value $\Phi_f = -2.5 kT_e/e$. This value is typical for small objects in astrophysical plasmas.