Repetition, Fys 5120; Advanced QFT

Lagrangians ("Tree-level", Fys 4170)

QFT contains Quantum fluctuations
- i.e. loops; quantum physics beyond
(NR-) Quant. Mech.

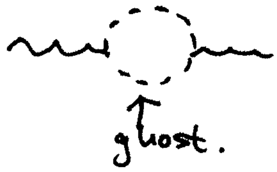
Archetypical loops (- QED)

* Fermion Self energy $\Sigma(p)$  + ...

* Boson Self. energ. (Vac. pol) $\Pi_{\mu\nu}(q)$  + ..

* Vertex correction $\Delta\Gamma^{\mu}(p',p)$ 

— In non-abelian theories also

× Vac. pol:  

× Vertex 

(QCD & elweak)

These loops need <u>renormalization</u>
Two-fold problem:

1) $|e\rangle$, $|\gamma\rangle$ Not stable (Need <u>renorm.</u>)
2) Loops are divergent; - Need <u>regularization</u>

Regularization: by cut-off (Λ)

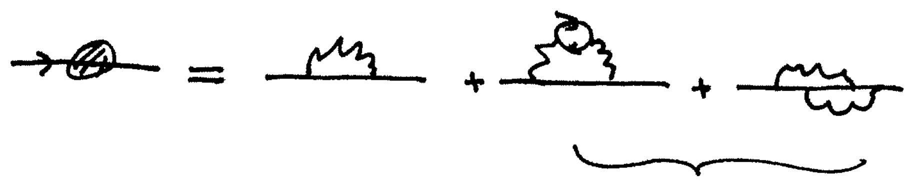
– or (most "modern") by <u>Dimension Regularization</u>

Mathematical tricks: "Feynman parametrization", "Wick rotation" (to complex plane, $p_0 \to -ip_4$)

Momentum integr. in Euclidian space

* Reg. described in lectures, P&S and M&S.
(– and Mid-Term exam !)

Ferm. Self-energy:
(irreducible)



Self energy $\Sigma(p)$ can be written
(NB! : <u>Ignoring (± 1), ($\pm i$) details</u>)

$$\Sigma(p) = A + (\gamma\cdot p - m)B + (\gamma\cdot p - m)^2 \tilde{\Sigma}_c(p)$$

A and B logaritm. divergent ($\sim \ln(\frac{\Lambda}{m_0})$)
– or $\sim \left(\frac{1}{4-d}\right)$ in dim. reg
(d = dim. of space)

Physical (renormalized) Mass : $m = m_0 + \delta m$
where $\delta m = A$; m_0 = "bare mass"
(–in Lagrangian)

The bare propagator:

$$S_0(p) = \frac{\gamma\cdot p + m_0}{p^2 - m_0^2} = \frac{1}{\gamma\cdot p - m_0}$$

The Modified (by loop correct.) propagator

$$S(p)_{Mod} = \frac{1}{(\gamma\cdot p - m_0) - \Sigma_1(p)} = \frac{1}{(\gamma\cdot p - m)(1-B) + (\gamma\cdot p - m)^2\tilde{\Sigma}_c}$$

Near the pole

$$S(p)_{Mod} \underset{\gamma\cdot p \to m}{\sim} \frac{(1+B)}{\gamma\cdot p - m} \overset{=}{?} \frac{Z_2}{\gamma\cdot p - m}$$

Renorm constant $Z_2 = 1 + B$

Renormalized Dirac (fermion) field $\psi_R = (Z_2)^{-1/2}\,\psi_0$ "bare field"

Renormalized self energy (Phys. $\Sigma(p)$):

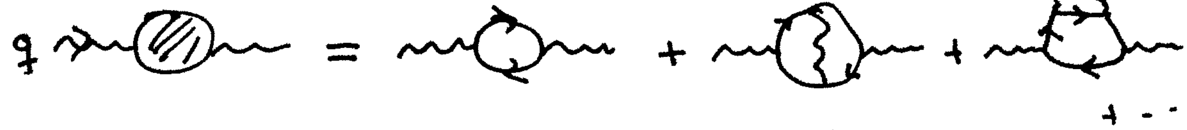
$$\Sigma(p)_R = (\gamma\cdot p - m)^2 \left[\tilde{\Sigma}_c(p)\right]_{\substack{m_0 \to m \\ e_0 \to e}}$$

For $|p^2| \gg m^2$

$$\Sigma_1(p)_R \simeq \frac{\alpha_{em}}{4\pi}\,\gamma\cdot p \ln\left[\frac{|p^2|}{m^2}\right]$$

$\sim$

Vac. Polarization



Not calculated here

$$\Pi_{\mu\nu}(q) = D g_{\mu\nu} + \hat{\Pi}(q^2)\left[g_{\mu\nu}q^2 - q_\mu q_\nu\right]$$

Cut-off reg $D \sim \Lambda^2$ (problematic)

Gauge invar / Current cons $\Rightarrow q^\mu \Pi_{\mu\nu}(q) = 0$

Dim-reg $D = 0$ directly

$\hat{\Pi}(q^2)$ = logarithm. divergent.

For Dim-reg

$$\underbrace{I_1 = -\frac{\Gamma(2)\cdot\Delta}{\Gamma(1)\,\Gamma(1-d/2)}\,I_2}\ ; \quad I_N = \int \frac{d^dp}{(p^2-\Delta)^N} = \frac{i(-i)^N\,\Gamma(N-\frac{d}{2})}{\Gamma(N)\,\Delta^{(N-d/2)}}$$

$$\Delta \equiv m_0^2 - x(1-x)q^2$$

$$\hat{\Pi}(q^2) = 2e_0^2 f(d)\int_0^1 dx\, x(1-x)\, I_2(\Delta)\ ; \quad f(d) = (\mathrm{Tr}\,\mathbb{1}_D)_d$$

Relation between I_1 & I_2 gives $D=0$ in dimreg.

Modified Photon propagator near pole

$$D^{\mu\nu}_{Mod} = \frac{-g^{\mu\nu}}{k^2}\cdot Z_3\ ; \quad Z_3 = (1-\hat{\pi}(0))^{-1} = 1+\hat{\pi}(0) + \mathcal{O}(e_0^4)$$

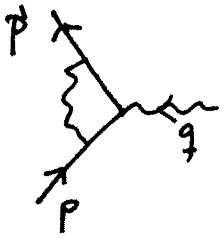
Renormalized field $A^R_\mu = Z_3^{-1/2} A^0_\mu$ (A^0_μ = bare em. field)

Renormalized $\hat{\Pi}(q^2)$:

$$[\hat{\Pi}(q^2)]_R = -2\,\frac{\alpha_{em}}{\pi}\int_0^1 dx\, x(1-x)\ln\left\{\frac{m^2}{m^2-x(1-x)q^2}\right\}$$

(- which is zero for $q^2=0$)

Vertex correction. Modified Vertex $\Gamma^\mu = \gamma^\mu + \Delta\Gamma^\mu$



$$\sim \Delta\Gamma^\mu(p',p) = \Delta F_1\cdot\gamma^\mu + \frac{F_2(q^2)}{2m}\, i\sigma^{\mu\nu} iq_\nu$$

(- have used Dirac eq. $\gamma\cdot p\, u(p) = m u(p)$
$\overline{u(p')}\,\gamma\cdot p' = \overline{u(p')}\cdot m$)

$$F_2(q^2=0) = \frac{\alpha_{em}}{2\pi} = \frac{1}{2}(g_e-2) \sim \text{anomalous magnetic moment}$$

$=$ <u>finite!</u> while <u>$\Delta F_1(q^2)$ diverges!</u>

Renorm.-factor for vertex:

$$Z_1 = [1+\Delta F_1(0)]^{-1} = 1-\Delta F_1(0) + \mathcal{O}(e_0^4)\ ; \quad \Gamma^\mu_{Mod}(q=0) = \gamma^\mu[1+\Delta F_1(0)]$$

Rep-5

Ward identity (1) & 2)) (vertex related to Self-energies)

1) $$q_\mu \left[\Delta\Gamma^\mu(p',p)\right] = \Sigma(p') - \Sigma(p)$$ (up to a sign!)

(both for Renormalized and Non-renorm. vertices

- Implies $$\frac{\partial \Sigma(p)}{\partial p_\mu} = \Delta\Gamma^\mu(p'=p, p)$$

- and $Z_1 = Z_2$.

The renormalized charge is

$$e = \frac{Z_2}{Z_1}\sqrt{Z_3}\, e_0 = e_0\sqrt{Z_3}$$

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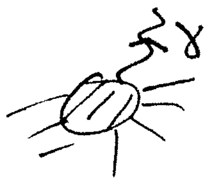
2)  $= \underbrace{\varepsilon_\mu(k)\, \mathcal{M}^\mu}$ = ampl.

$\Rightarrow$ $\underbrace{k_\mu \mathcal{M}^\mu = 0}$ (Gauge-invariance)

Infra-Red (IR) property of $F_1(q^2)_R$ (Renorm.) :

$$\underbrace{F_1(q^2)_R = 1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \ln\left(\frac{Q^2}{\mu^2}\right)} \leftrightarrow$$

μ = IR cut-off (to get ~~def~~ IR def)

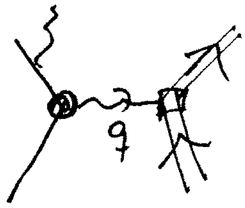
$$Q^2 = \begin{cases} -q^2 & \text{for } |q^2| \gg m^2 \\ m^2 & \text{for } m^2 \gg q^2 \end{cases}$$

["Sudakov log.'s"]

$$f_{IR}(q^2) \approx \ln(-q^2/m^2) \text{ for } |q^2| \gg m^2$$

Take into account also soft bremsstrahlung :

Rep 6



$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Measured}} =$$

$$\left(\frac{d\sigma}{d\Omega}\right)(\underbrace{e\to e}_{El}) + \frac{d\sigma}{d\Omega}\left[e\to e+\gamma\,(\omega < E_m)\right]$$

$$= \left(\frac{d\sigma}{d\Omega}\right)_{El}\left\{1-\frac{\alpha}{\pi}\, f_{IR}(q^2)\,\ln\left(\frac{Q^2}{E_m^2}\right)\right\}$$

(exponentiated for many soft γ's)

Imaginary Parts

has imag. part for $q^2 \ge (2m)^2$
(Production of real pair possible

$2\,\mathrm{Im}(\quad) = \int d\Pi\, |\quad|^2$

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Optical theorem:

$\mathrm{Im}\, \mathcal{M}(a\to a) = 2E_{CM}P_{CM}\,\sigma_{Tot}(a\to \text{"Anything"})$

Unstable Particles (Propagator)

(Boson) $\dfrac{1}{p^2-m_0^2\mp\Sigma(p^2)} = \dfrac{1}{p^2-m^2-i\left[\mathrm{Im}\,\Sigma_R(p^2)\right]} \to \dfrac{1}{p^2-m^2+im\Gamma}$

(?)

Breit-Wigner form

$m^2-m_0^2-\mathrm{Re}\,\Sigma(m^2)=0$

W-prop: $+ \quad + \cdots$

Dimensions in Dim. Reg.:

$S = \int d^d x \, \mathcal{L}$;

$\mathcal{L}_D = \bar{\Psi}(i\gamma\cdot\partial - m)\psi$; $\mathcal{L}_{int} = e\bar{\Psi}\gamma^\mu\psi A_\mu$

$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\mathrm{Dim}(S) = 0$, $\mathrm{Dim}(x) = -1$, $\mathrm{Dim}(m) \equiv +1$

$\mathrm{Dim}(\partial_\mu) = \mathrm{Dim}(m) = +1$, $\mathrm{Dim}(\psi) = \frac{d-1}{2}$

$\mathrm{Dim}(A_\mu) = \frac{d}{2} - 1 \Leftarrow \mathrm{Dim}(F_{\mu\nu}) = \frac{d}{2}$

$\mathrm{Dim}(e) = (2 - \frac{d}{2})$, $\mathrm{Dim}(\mathcal{L}) = d$

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$\boxed{\text{Renormalization "Group}^{(?)}\text{" Equations (RGE) (- or: Callan-Symanzik eqs)}}$ ϕ^4 Theory

$$\mu\frac{d\Gamma_0^{(N)}}{d\mu} = \left\{\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + N\gamma(\lambda)\right\}\Gamma^{(N)}_{Ren}(p_i, \lambda, \mu) = 0$$

$\Gamma^{(N)} = ?$  $G^{(N)}(x_i) \rightarrow$ Momentum space $\Gamma^{(N)}(p_i)$

Divide with propagators $\Gamma^{(2)}(p)$

$\boxed{\beta(\lambda) \equiv \mu\frac{\partial\lambda}{\partial\mu}}$, $\boxed{\gamma(\lambda) = -\mu\frac{\partial}{\partial\mu}(\ln\sqrt{Z_\phi})}$

($\Gamma_0^{(N)}$ indep. of μ, but dep. on Λ)

QCD for massless fermions

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + N_F\gamma_F(g) + N_B\gamma_B(g)\right\}\Gamma^{(N_F,N_B)}_{Ren}(p_i, g, \mu) = 0$$

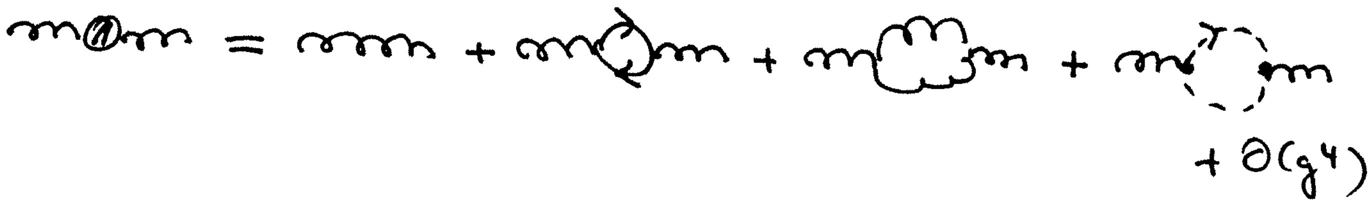
($m \neq 0$ Extra term $\sim \gamma_m(g)\, m\frac{\partial}{\partial m}$)

Renorm inverse bosonpropagator Rep 8

(Lead. Log approx.!)

$$\left[\Gamma^{(0,2)}(p)\right]^{\alpha\beta} = (p^\alpha p^\beta - g^{\alpha\beta}p^2)\left\{1 + d_B\, g^2 \ln\left(-\frac{p^2}{\mu^2}\right) + \mathcal{O}(g^4)\right\}$$

$d_B = \frac{1}{16\pi^2}\left[\frac{13}{2} - \frac{2}{3}N_f\right]$ ($|p^2| \gg \mu^2$; constant terms neglected)



Inverse fermionpropagator (renorm.)

$$\Gamma^{(2,0)}(p) = \gamma\cdot p\left\{1 + d_F \ln\left(-\frac{p^2}{\mu^2}\right) + \mathcal{O}(g^4)\right\}$$

(Landau gauge $d_F = 0$)

Vertex - renormalized

$$\Gamma^{(2,1)}(p',p)^{\mu,a}_{q=0} = g\gamma^\mu t^a\left\{1 + d_V \ln\left(-\frac{p^2}{\mu^2}\right) + \mathcal{O}(g^4)\right\}$$

Beta - funct.

$$\beta = (2d_V - d_B)g^3 = \frac{-1}{16\pi^2}\hat{b}\cdot g^3 + \mathcal{O}(g^5)$$

$+\mathcal{O}(g^5)$

$$\hat{b} = 11 - \frac{2}{3}N_f \equiv 16\pi^2 b$$

Eff coupling $\bar{g}(t)$: $t = \frac{1}{2}\ln\frac{Q^2}{\mu^2}$ $(Q = \mu e^t)$

$$\frac{\partial \bar{g}}{\partial t} = \beta(\bar{g}) \Rightarrow (\bar{g})^2 = \frac{g^2}{1 + 2b g^2 \cdot t}$$

This can also be written

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{4\pi}\hat{b}\ln\frac{Q^2}{\mu^2}} \equiv \frac{4\pi}{\hat{b}\ln\left(\frac{Q^2}{\Lambda^2_{QCD}}\right)}$$

Exper: $\Lambda_{QCD} \sim$ few hundred MeV

(changes when higher orders taken into account!)

Typical factor from RGE solutions

$$\exp\left\{\int_0^t dt'\,\gamma_B(\bar{g}(t'))\right\} = \left(1 + 2bg^2t\right)^{\frac{d_B}{2b}} = \underline{\underline{\left[\frac{\alpha_s(\mu^2)}{\alpha_s(Q^2)}\right]^{\frac{d_B}{2b}}}}$$

Example

$$\left[\Gamma^{(0,2)}(p)\right]^{\alpha\beta}_{RGE} = \left(p^\alpha p^\beta - g^{\alpha\beta}p^2\right)\left\{1 + b\left[g(\mu^2)\right]^2\ln\left(-\frac{p^2}{\mu^2}\right)\right\}^{\frac{d_B}{2b}}$$

for $\frac{|p|^2}{\mu^2} \gg 1$

Explicite calculations of typical loops in QCD:

P&S pp 521–531

(M&S: pp 357–362)

The Axial (triangle) anomaly

This anomaly is a tricky thing. There are two things to remember:

1) The process $\pi^0 \to 2\gamma$
2) The (induced?) vertex $Z \to 2\gamma$

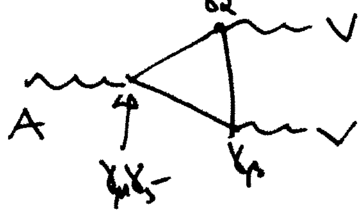
1) It can be shown that — in the precence of electromagnetic interactions — the divergence of the axial current is not zero in the limit $m_q \to 0$, but

$$\partial_\mu j^\mu_A = -\frac{e^2}{16\pi^2}\,\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

(eq. 19.45 in P&S — the derivation i.e. p660 is outside the scope of this course)

As pion decay is given by matrix elements of the axial current, this relation determine the amplitude for $\pi^0 \to 2\gamma$.

In perturbation theory the diagram

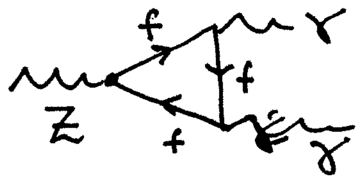


is ambigous — depending on regularization procedure

In dim-reg. the loop integration seems log. divergent, being proportional to $\Gamma(2-\frac{d}{2})$. But then the γ_μ-algebra gives a factor $\sim (4-d)$, making the amplitude finite ... – and in accordance with the anomaly result (P&S 661-664)

2) A vertex for $Z \to \gamma\gamma$ does not exist in the SM. Therefore, there is no place to hide a divergence in

(no parameter!)



Therefore, if one sum over all fermions (allowed by SM couplings) in the loop one should get zero !

("Anomaly cancellation")

– otherwise renormalizability would be destroyed !

For instance, anomaly cancellation require for the electric charges Q in each generation/family:

$$(Q_{\nu_e} + Q_e) + N_f(Q_u + Q_d) = 0 \quad ; N_f = 3$$

(– and sim for 2. and 3. gen.)

– i.e. "charge quantization"

Some

Unphysical Higgs couplings

Let us consider the Higgs (couplings) Lagrangian for the third generation of quarks:

$$q_L^{(3)} = \begin{pmatrix} t \\ b \end{pmatrix}_L \quad ; \quad \overline{q_L^{(3)}} = (\bar{t}_L, \bar{b}_L) :$$

In addition to the Higgs field $\phi = \begin{bmatrix} \phi_+ \\ \phi_0 \end{bmatrix}$, one needs also $\tilde{\phi} = \begin{bmatrix} \phi_0^\dagger \\ -\phi_+^\dagger \end{bmatrix} \equiv \begin{bmatrix} \phi_0^\dagger \\ -\phi_- \end{bmatrix}$:

$$\mathcal{L}_{3q\phi} = -G_b \left[\bar{q}_L^{(3)} b_R \phi + \phi^\dagger \bar{b}_R q_L^{(3)} \right] - G_t \left[\bar{q}_L^{(3)} t_R \tilde{\phi} + \tilde{\phi}^\dagger t_R q_L^{(3)} \right]$$

We assume SSB, and ($\Rightarrow$) $\langle \phi_0 \rangle_{vac} = \hat{v} \equiv \frac{v}{\sqrt{2}}$

~~and~~ In unitary gauge, 3 deg. of freedom ~~is~~ are transf. away: $\phi \to \phi' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+H \end{pmatrix}$ where H is the physical Higgs field.

This is not done here, we just write:

$$\phi = \begin{bmatrix} \phi_+ \\ \hat{v} + \Delta\phi_0 \end{bmatrix} \quad ; \quad \tilde{\phi} = \begin{bmatrix} \hat{v} + (\Delta\phi_0)^\dagger \\ -\phi_- \end{bmatrix}$$

In any case, we obtain the mass part:

$$\mathcal{L}_{3q\,Mass} = -G_b \hat{v} (\bar{b}_L b_R + \bar{b}_R b_L) - G_t \hat{v} (\bar{t}_L t_R + \bar{t}_R t_L)$$

The couplings to the charged unphys. Higgs fields are given by

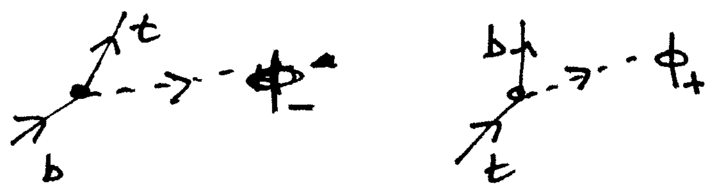
$$-\mathcal{L}_{sq}(\phi_\pm) = G_b\{\phi_+ \bar{t}_L b_R + \phi_- \bar{b}_R t_L\}$$
$$- G_t\{\phi_- \bar{b}_L t_R + \phi_+ t_R b_L\}$$

With no quark mixing: $m_b = G_b \hat{v}$ & $m_t = G_t \hat{v}$

or $$G_b = \frac{g_W}{M_W} m_b \quad \& \quad G_t = \frac{g_W}{M_W} m_t$$

(where we have used $M_W = \frac{1}{2} g v$ and $g_W = \frac{g}{\sqrt{2}}$)

From this we may obtain the couplings



(cfr Fig 21.5, p749 in P&S)

BUT: We have also quark mixing, and for inst. $d \leftrightarrow t$ transitions for $\phi_\pm$ exchange.

This can be obtained by the substitution

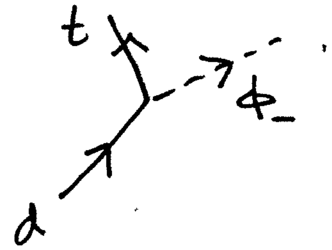
$$G_b = \left(\frac{g_W}{M_W} m_b\right) \longrightarrow \frac{g_W}{M_W} m_d V_{td}$$

where V_{td} is the relevant CKM matrix element. - and we obtain the interaction Lagrangian

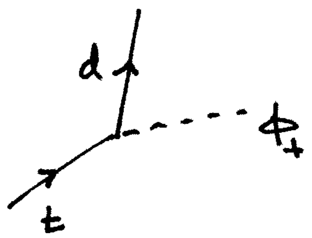
$$-\mathcal{L}_{3q\phi}(\phi_{\pm} d t) =$$

$$\frac{g_W}{M_W}\left\{ \bar{t}\,\phi_+ V_{td}(m_d R - m_t L)\,d + \bar{d}\,\phi_- V_{td}^{*}(m_d L - m_t R)\,t \right\}$$

giving the couplings (sim. for $t \leftrightarrow s$ with $m_d V_{td} \rightarrow m_s V_{ts}$)



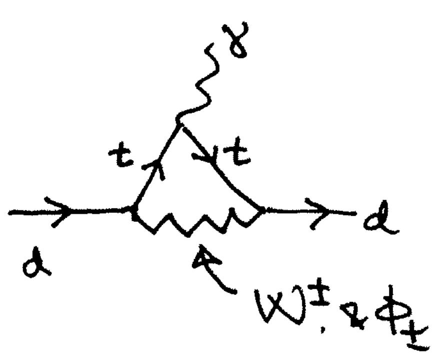
$$-iV_{td}\cdot\frac{g_W}{M_W}(m_d R - m_t L)$$



$$-iV_{td}^{*}\cdot\frac{g_W}{M_W}(m_d L - m_t R)$$

- for insert to be used in weak $d \to d\gamma$

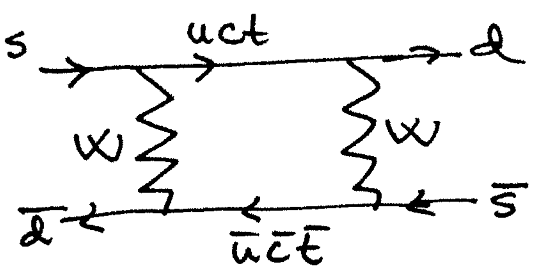
if Feyn. gauge $\frac{-g_{\mu\nu}}{k^2 - M_W^2}$ is used for $W^{\pm}$



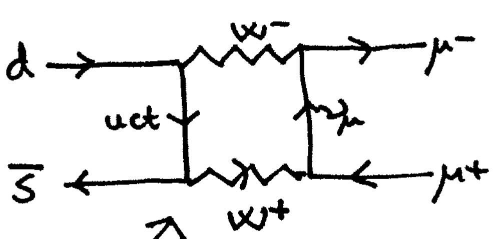
W's in loops

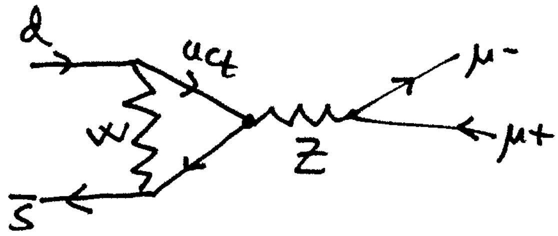
Examples:

$K^0 \leftrightarrow \bar{K}^0$ mixing



and $K^0 \to \mu^+\mu^-$





(Here) one might(?) expect amplitude $\sim \dfrac{g_W^4}{(M_W^2)^2}$

However, dimension analysis shows that the $d\bar{s} \to \mu^+\mu^-$ amplitude is $\sim \dfrac{g_W^4}{(\text{Mass})^2}$

BUT: Here the "GIM"-mechanism due to quark mixing must be taken into account

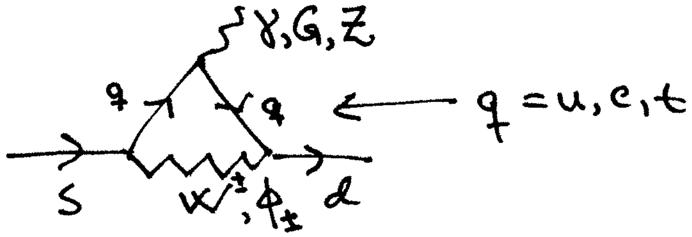
The CKM-matrix V should be unitary $V^\dagger V = \mathbb{1}$

- or $(V^\dagger V)_{ij} = \delta_{ij} \Rightarrow V^\dagger_{ik} V_{kj} = \delta_{ij} \Leftrightarrow V^*_{ki} V_{kj} = \delta_{ij}$

For $s \leftrightarrow d$ transitions with $q = u, c, t$ in a loop, we thereby get:

$$\sum_{q=uct} \lambda_q = 0 \quad \text{for} \quad \lambda_q \equiv V^*_{qd} V_{qs}$$

Many loop amplitudes, like $d\bar{s} \to \mu^+\mu^-$ — and the "Penguin diagram"



will therefore have the structure

$$\sum_{q=uct} \lambda_q F(m_q) = \lambda_t [F(m_t) - F(m_c)] + \lambda_u [F(m_u) - F(m_c)]$$

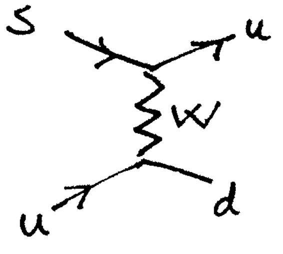
which would be zero if the quark masses m_u, m_c, m_t were equal.

In any case it means that the quark-mass-independent part of the loop amplitudes (— some times the dominant ones) cancel!

(See e.g. "Problems, April 2013" — PA — pages 1-4)

QCD and Non-leptonic decays:

— $K \to \pi\pi$ -decays (— or sim. $D \to K\pi, \pi\pi$ and $B \to D\pi, DK, K\pi, \pi\pi$) are given at quark level as the product of two left-handed quark currents —

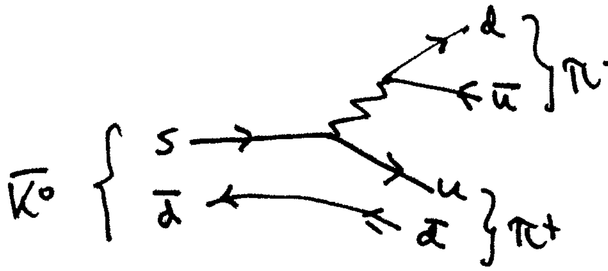


to lowest order in perturbative QCD:

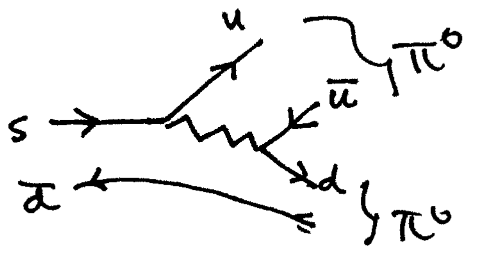
$$\mathcal{L}^{(0)}_{weak} = -4 \frac{G_F}{\sqrt{2}} V_{us} V^{*}_{ud}\, Q_A$$

$$Q_A \equiv (\bar{u} \gamma^{\mu} L s)(\bar{d} \gamma_{\mu} L u)$$

where the physical processes are illustrated like

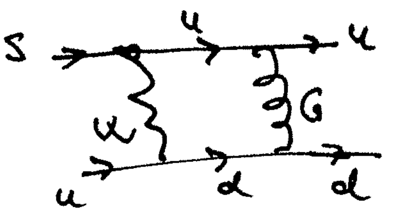


("Facturizable")



(Non-factorizable = "Color suppressed")

There are QCD-corrections (pert.) to Q_A, like (and crossed W-G) giving amplitudes (in leading log. approx.):



$$Q_A \rightarrow \left(\frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2}\right)\left[f_A Q_A + f_B Q_B\right]$$

where $f_{A,B}$ are simple numbers of order one, and μ is a renormalization point $\mu \sim M_K$ or $\mu \sim 1\,\text{GeV}$, say
(P&S pp 605-612)

Q_B is another quark operator generated by perturbative QCD:

$$Q_B = (\bar{d}\gamma^\mu L s)(\bar{u}\gamma_\mu L u)$$

To obtain this: Use loop techniques/integrals, Dirac algebra $\gamma^\mu\gamma^\nu\gamma^\sigma = g^{\mu\nu}\gamma^\sigma - g^{\mu\sigma}\gamma^\nu + g^{\nu\sigma}\gamma^\mu + i\varepsilon^{\mu\nu\sigma\lambda}\gamma_\lambda\gamma_5$,

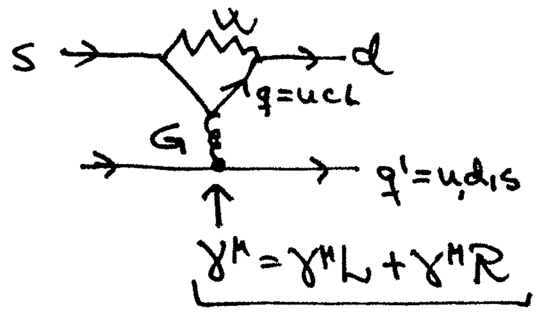
color matrix relation:

$$2(t^a)_{in}(t^a_{lj}) = \delta_{ij}\,\delta_{ln} - \frac{1}{N_c}\delta_{in}\delta_{lj} \, ,$$

and Fierz transformation

$$\bar{\Psi}_A \gamma^\mu L \Psi_B \; \bar{\Psi}_C \gamma_\mu L \Psi_D = \bar{\Psi}_A \gamma^\mu L \Psi_D \; \bar{\Psi}_C \gamma_\lambda L \Psi_B$$

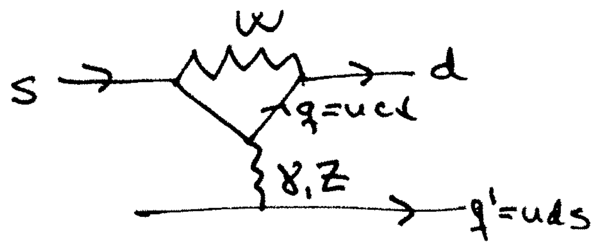
The QCD penguin diagram



Gives additional four new operators

and "electroweak penguins" even 4 more

$\Downarrow$



$$\mathcal{L}_{weak} = -4\frac{G_F}{\sqrt{2}}(KM\text{ fact})\sum_i^{10} c_i Q_i$$

The coefficients c_i satisfy QCD-RGE equations!

BUT:

Matrix elements $\langle \pi\pi | Q_i | K \rangle$ are difficult "confinement-problems"

In some cases "Factorization"

i.e. $\langle \pi\pi | Q_i | K \rangle \sim \langle \pi | j_\mu^{w}(1) | 0 \rangle \langle \pi | j_w^{\mu}(2) | K \rangle$ *)

Might work (– especially for $B \to 2\pi$, where π's are energetic) –

– but in general not .....

(Lattice gauge theory or Quark Models used....)

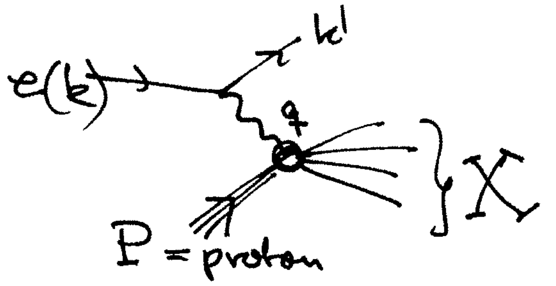
(*) Matrix elements of currents are reasonably well known)

Rep 20

Deep Inelastic Scattering (DIS)

Played crucial role to understand quark picture (1967 $\longrightarrow$ .....)

$ep \to eX$; $X =$ "Anything"

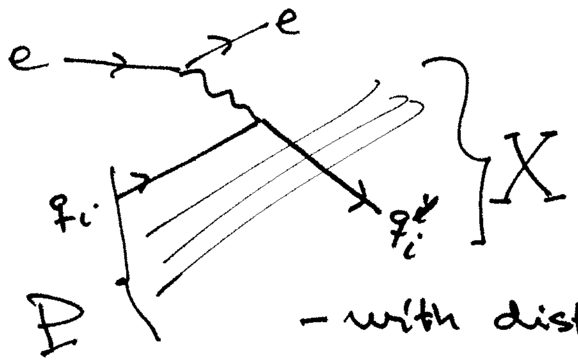


Amplitude

$$\mathcal{M} \sim j^\mu(e) \cdot \frac{e^2}{q^2} \cdot J_\mu(P \to X)$$

$$j^\mu(e) \equiv \bar{u}(k')\gamma^\mu u(k) \; ; \; J_\mu(P \to X) = \langle X | J_\mu^{em} | P \rangle$$

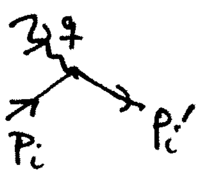
Parton/Quark Picture: $\sum_X |X\rangle\langle X| = \mathbb{1}$



Proton contains "partons" (= quarks, gluons(?!))

– with distribution functions $f_i(x)$

$$\left(\frac{d^3\sigma}{d\Omega dE'}\right)(eP \to eX) = \sum_i \int_0^1 dx \, f_i(x) \left[\frac{d^3\sigma}{d\Omega dE'}\right](eq_i \to eq_i')$$



$p_i = x_i P$

$p_i' = q + p_i$

$\Rightarrow x_i = -\frac{q^2}{2P \cdot q}$; P = proton momentum

x = Momentum fraction of scattered "parton"

A well founded description of DIS is given in P&S, pp 623–627.

The bottom line(s) are/(is) the following:

* The scattering cross section can be written in terms of functions $W_1(Q^2, \nu)$ and $W_2(Q^2, \nu)$ related to the tensor $[Q_2 \equiv -q^2 ; \nu \equiv P\cdot q/M_P]$

$$W^{\mu\nu} \equiv i \int d^4x \, e^{iq\cdot x} \langle P | J^\mu_{em}(x) J^\nu_{em}(0) | P \rangle,$$

where $J^\mu_{em}(x)$ is the electromagnetic current operator

$W_{1,2}$ are – experimentally – exhibiting "Bjorken scaling":

They seem to depend only on $x = \frac{Q^2}{2M\nu}$ – and not on Q^2 and ν separately.

Theoretically from QCD: Scaling is valid for $\alpha_s \to 0$, while Scaling Violations can be calculated within a RGE framework

* The distribution functions for quarks, antiquarks <u>and gluons</u> within the proton (–at high momentum) can be indirectly measured from the cross section (–most recently measured at HERA experiment at DESY in Hamburg). They are important input for the present LHC experiment at CERN.

SSB in QCD

Consider QCD for the lightest 3 (ev. 2) quarks:

$$\mathcal{L}^{(3)}_{QCD} = \bar{q}\{i\gamma\cdot D - \mathcal{M}_q\}q + \mathcal{L}_G \; ; \; (\mathcal{L}_G = -\tfrac{1}{4}G^2 + \mathcal{L}_{ghost})$$

where $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$ and $\mathcal{M}_q = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}$

We split the quark fields in left- and right-handed parts $\underline{q = q_L + q_R}$

The Lagrangian may then be written

$$\mathcal{L}^{(3)}_{QCD} = \bar{q}_L i\gamma\cdot D q_L + \bar{q}_R i\gamma\cdot D q_R - \bar{q}_L \mathcal{M}_q q_R - \bar{q}_R \mathcal{M}_q q_L + \mathcal{L}_G$$

In the limit $\mathcal{M}_q \to 0$, this Lagrangian has a

<u>Global</u> $SU(3)_L \times SU(3)_R$ symmetry:

$\updownarrow$

(i.e. <u>not</u> a gauge symmetry!) The Lagrangian is sym. under the independent transf:

$q_L \to q_L' = V_L q_L \; ; \; \bar{q}_L' = \bar{q}_L V_L^\dagger$ and sim. for R sector

$q_R \to q_R' = V_R q_R \; ; \; \bar{q}_R' = q_R V_R^\dagger$

In practise, the limit $m_q \to 0$ is much better for $q = u, d$ (having current quark masses less than 10MeV) than for $q = s$, having $m_s \sim 100$ to 150 MeV. So in some cases, one restrict oneself to discussion of global $SU(2)_L \times SU(2)_R$ for $q = \binom{u}{d}$. The point is that the two (- or three) lightest quarks have masses much lighter than a typical hadronic mass like $m_\rho \sim 800$MeV or $m_p \sim 940$ MeV.

NB! The generators of global $SU(N)_{L,R}$ (for N=2,3) commutes with the generators of QCD (i.e. $SU(3)_c$) !

In (perturbative) QED $\langle 0|\bar{\psi}\psi|0\rangle = 0$

On the contrary, in QCD, with (very) strong confinement one has a "quark condensate":

$$\langle 0|\bar{q}q|0\rangle \equiv \langle 0|(\bar{q}_L q_R + \bar{q}_R q_L)|0\rangle \simeq (-240\text{MeV})^3 \cdot N$$

This condensate breaks the global $SU(3)_L \times SU(3)_R$ sym. down to $SU(3)_V$ (pp 667-670 in P&S)

The 8 generators of $SU(3)_L$ and the 8 gen. of $SU(3)_R$ may be combined to 8 gen. of $SU(3)_V$ (V = vector) and 8 gen. of $SU(3)_A$ (A = axial vector)

The quark condensate breaks the 8 generators of $SU(3)_A$ and there will be – in the limit $M_q \to 0$ (i.e. $m_q \to 0$ for $q = u, d, s$) 8 Goldstone bosons, the pseudo-scalar bosons ($\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta_8$)

For the $SU(2)_L \times SU(2)_R \to SU(2)_V$ case ($m_s \to 0$ excluded) there will be 3 Goldstones ($\pi^\pm, \pi^0$). BUT, the $SU(N)_L \times SU(N)_R$ sym. (for N=2,3) is not exact, and therefore the ("would be") Goldstones have a "small" mass!

Consider the axial current

$$j_A^{\mu,a} = \bar{q}\, T^a \gamma^\mu \gamma_5\, q \qquad \left(\begin{array}{l} a = 1,2,3 \text{ for } N=2 \\ a = 1,\dots 8 \text{ for } N=3 \\ T^a = \text{gen. for } SU(N) \end{array} \right)$$

Using the Dirac equation, one finds

$$i\partial_\mu j_A^{\mu,a} = -\bar{q}\, \{T^a, M_q\}\, \gamma_5\, q$$

($\{A,B\} \equiv AB + BA$)

On the other hand, the axial current has the matrix element

$$\underline{\langle 0 | j_A^{\mu, a} | \pi^b(k) \rangle = -i k^\mu f_\pi \delta^{ab}}$$

($a, b = 1, 2, 3$ for $N = 2$; $a, b = 1, \cdots 8$ for $N = 3$)

(Ideally $\underbrace{f_\pi = f_K}_{SU(3)\text{-sym.}}$; but exper. $f_K / f_\pi \sim 1.2$)

Using these equations for $\pi^\pm$ would give

$$(m_u + m_d) \langle 0 | \bar{q} \gamma_5 q | \pi^\pm \rangle = m_{\pi^\pm}^2 f_\pi$$

$(q = u, d)$

Here $(\bar{q} \gamma_5 q)$ has dim $= 3$ and $|\pi^\pm\rangle$ has dim $= -1$

Writing $\langle 0 | \bar{q} \gamma_5 q | \pi^\pm \rangle = M^2$,

we have
$$\boxed{(m_{\pi^\pm})^2 = \frac{m_u + m_d}{f_\pi} M^2}$$

f_π

It can be shown that

$$\langle 0 | \bar{q} q | 0 \rangle = -f_\pi M^2$$

(for $q = u$ or $q = d$ or $q = s$)

such that
$$\underline{(m_{\pi^\pm})^2 = -\frac{(m_u + m_d)}{f_\pi^2} \langle 0 | \bar{q} q | 0 \rangle}$$

Similarly (for SU(3)),

(one flavour)

$$\underline{(m_{K^\pm})^2 = -\frac{(m_u + m_s)}{f_\pi^2} \langle 0 | \bar{q} q | 0 \rangle}$$

($q = u, d,$ or s)

Path Integral Formalism

For the <u>quantum mechanical case</u> this is described in P&S pp 275-282 and lecture notes of Fys4110 pp 20-24

For (Quantum) Field Theory (– when coordinates are replaced by fields) the theory can be described in terms of generating functionals Z, in the simplest case for a (real) scalar theory:

$$\boxed{Z[J] = \int \mathcal{D}\phi \, \exp\left\{i \int d^4x \left[\mathcal{L}_\phi + \phi(x) J(x)\right]\right\}}$$

where J is a "source"

For the Dirac case, similarly

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \, \exp\left\{i \int d^4x \left[\mathcal{L}_D + \bar{\eta}\Psi + \bar{\Psi}\eta\right]\right\},$$

where $\eta, \bar{\eta}$ are sources. Here $\eta, \bar{\eta}$ are anticommuting quantities; – i.e. Grassman variables.

Taking functional derivatives with resp. to the sources, one obtain the fields ---, such that

$$\left[\left(-i \frac{\delta}{\delta J(x_1)}\right)\left(-i \frac{\delta}{\delta J(x_2)}\right) Z[J] \right]_{J=0}$$

$$= \langle 0 | T(\phi(x_1)\phi(x_2)) | 0 \rangle \cdot \left(Z[J=0] \right),$$

and similarly

$$\left[\left(-i \frac{\delta}{\delta \bar{\eta}(x_1)_\alpha}\right)\left(+i \frac{\delta}{\delta \eta(x_2)_\beta}\right) Z[\bar{\eta}, \eta] \right]_{\bar{\eta}=0, \eta=0}$$

$$= \langle 0 | T(\psi(x_1)_\alpha \bar{\psi}(x_2)_\beta) | 0 \rangle \cdot \left[Z[\eta=0, \bar{\eta}=0] \right]$$

Generally, various N-point functions may be obtained by taking functional derivatives with respect to the respective sources

An important property of the Grassman variables θ_i is

$$\underbrace{\prod_i \int d\theta_i^* d\theta_i \, e^{-\theta_i^* B_{ij} \theta_j} = \det B}_{},$$

which will be used later to define the ghost fields.

Path integrals are convenient in describing the Gauge-fixing problems

Gauge-fixing for the U(1) gauge field (— like the elmagn. field):

One considers the functional integral

$$Z_A = \int \mathcal{D}A \, e^{i S[A]} \quad , \text{ where } S \text{ is the action}$$

$$S = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu}\right)$$

$$= \tfrac{1}{2} \int d^4x \, A_\mu \left[g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu \right] A_\nu \quad + \left(\begin{matrix}\text{surface}\\ \text{terms}\end{matrix}\right)$$

The operator $[g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]$ should correspond to the inverse propagator but, it is singular and the propagator itself is not well defined — due to gauge-invariance (Many gauges are physically equivalent),

$$A_\mu \to A'_\mu = A^{(\alpha)}_\mu = A_\mu + \tfrac{1}{e}\partial_\mu \alpha$$

The Fadeev-Popov method of Gauge-fixing uses the following trick:

Let $G(A) = 0$ be a gauge-fixing condition, for inst $G = \partial_\mu A^\mu$ or $G = \partial_\mu A^\mu - \omega(x)$, where $\omega(x)$ is any scalar function

Then we may multiply Z_A by

$$1 = \int \mathcal{D}\alpha \, \delta\left(G(A^{(\alpha)})\right) \det\left(\frac{\delta G(A^{(\alpha)})}{\delta \alpha} \right)$$

Further, — one integrates over all $\omega(x)$ with Gaussian weight close to $\omega = 0$, i.e. in total

$$Z_A \to N(\xi)\exp\left\{-i\int d^4x\,\frac{\omega^2}{2\xi}\right\}\cdot\int \mathcal{D}\alpha$$

ξ = arbitr. const

$$\cdot\int \mathcal{D}A\; e^{iS[A]}\;\det\left(\frac{\delta G(A^{(\alpha)})}{\delta\alpha}\right)\delta(\underbrace{\partial^\mu A_\mu^{(\alpha)}-\omega}_{G^{(\alpha)}})$$

Now, in the U(1) case $\dfrac{\delta G(A^{(\alpha)})}{\delta\alpha}=\dfrac{\partial^2}{e}$

$\Rightarrow \det\left(\dfrac{\delta G(A^{\alpha})}{\delta\alpha}\right)=\det\left(\dfrac{\partial^2}{e}\right)$ which is indep. of A_μ!

and can be put outside $\int\mathcal{D}A$

and the functional integral becomes

$$N(\xi)\det\left(\frac{\partial^2}{e}\right)\int\mathcal{D}\alpha\int\mathcal{D}A\;e^{i\left[S[A]-\frac{1}{2\xi}\int(\partial_\mu A^\mu)^2 d^4x\right]}$$

Now

$$S[A]-\frac{1}{2\xi}\int d^4x\,(\partial_\mu A^\mu)^2$$

$$=\frac{1}{2}\int d^4x\,A_\mu\left[g^{\mu\nu}\partial^2-\left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\nu$$

The propagator in momentum space $D_F^{\mu\nu}(k)$ must now satisfy the equation

$$\left[-k^2g^{\mu\nu}+\left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]D^F_{\nu\sigma}(k)=g^\mu{}_\sigma$$

$$\Rightarrow D^F_{\nu\sigma}(k)=\frac{1}{k^2+i\varepsilon}\left(-g_{\nu\sigma}+(1-\xi)\frac{k_\nu k_\sigma}{k^2}\right)$$

This trick is generalized to the Non-abelian case (- like $SU(3)_c$)

The difference from the Abelian case is now the gauge transf:

$$A_\mu^a \to (A_\mu^a)' \equiv (A_\mu^a)^{(\alpha)} = A_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b \alpha^c$$

(for small α's)

$$= A_\mu^a + \frac{1}{g}(D_\mu \alpha)^a$$

This covariant derivative is for the color octet field

$$(D_\mu \alpha)^a = (\delta^{ab}\partial_\mu + ig\, T^d_{ab} A^d_\mu)\alpha^b$$

$$T^d_{ab} \equiv (T^d)^{ab} = -i f^{dab}$$ (T = generators in 8 of SU(3))

Now $\det\left(\frac{\delta G^{(\alpha)}}{\delta \alpha}\right) = \det\left(\frac{1}{g}\partial^\mu D_\mu\right)$

which is not indep of A^a_μ.

Using the mentioned integral for Grassman variables this determinant gives an extra term (ghost-term) in the Lagrangian

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu)$$

$$+ \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b + \bar{\Psi}_k [i\gamma^\mu D^{kl}_\mu - m\,\delta_{kl}]\Psi_l$$

where $c, \bar{c}$ are Grassman 8 fields coupling only to A^c_μ throug the D^{ab}_μ term

($D^{ab}_\mu = \delta^{ab}\partial_\mu - g_s f^{abc} A^c_\mu$)

Within Higgs-based Lagrangians there is a ghost even for the simple U(1) Higgs toy model

$$\mathcal{L} = (iD^\sigma\phi)^*(iD_\sigma\phi) - V(\phi) - \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}.$$

With SSB the Higgs field can be written

$$\phi = \frac{1}{\sqrt{2}}\left[(v+H) + i\varphi\right]$$

NB: The imaginary part is not transformed away. For a gauge transf

$$A_\mu \to A'_\mu \equiv A_\mu^{(\alpha)} = A_\mu - \frac{1}{e}\partial_\mu\alpha,$$

the transf on ϕ is (for small α's)

$$\phi \to \phi' \equiv \phi^{(\alpha)} = e^{i\alpha}\phi = (1+i\alpha)[v+H+i\varphi]\frac{1}{\sqrt{2}}$$

i.e. $$\phi^{(\alpha)} = \frac{1}{\sqrt{2}}\left\{\left[(v+H)-\alpha\varphi\right] + i\left[\varphi + \alpha(v+H)\right]\right\}$$

One chooses a gauge fixing term:

$$G = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^\mu - \xi e v\varphi\right) - \omega,$$

which gives $$\frac{\delta G}{\delta\alpha} = \frac{1}{e\sqrt{\xi}}\left[-\partial^2 + \xi(ev)^2\left(1+\frac{H}{v}\right)\right],$$

leading to the ghost term (force)

$$\mathcal{L}_{Ghost} = \bar{c}\left[-\partial^2 - \xi M_A^2\left(1+\frac{H}{v}\right)\right]c$$

with $M_A = ev$. The vectorboson prop.

$$D_{\mu\nu}(k) = \frac{1}{k^2-M_A^2}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2-\xi M_A^2}(1-\xi)\right]$$

The φ and c-propagators are both: $\dfrac{1}{k^2-\xi M_A^2}$

(ξ = gauge param.)

For SSB due to Higgs in $SU(2)_L \times U(1)_Y$, formulae are the same, but with M_A^2 replaced by the 4×4 mass matrix for W_μ^i $(i=1,2,3)$ and B_μ ($U(1)$ field)

Thermal Field Theory

QFT may be used for Thermal - or "Finite temperature Field Theory". This is an euclidian theory, Namely: Time $t \to i\beta$, where $\beta = \frac{1}{kT}$, $T =$ (abs.) temperature.

β is periodic for bosons and anti-periodic for fermions.

$$p_0 \to i\omega_n \; ; \quad \omega_n = \begin{cases} 2\pi T \cdot n & \text{(bosons)} \\ 2\pi (n+\frac{1}{2}) T & \text{(fermions)} \end{cases}$$

Minkowski $$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} \xrightarrow[\text{Eucl. Temp QFT}]{} -\frac{i}{2\pi} \sum_{\omega_n} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\omega_n^2 + (\vec{p}^2 + m^2)}$$

Sum over frequencies ω_n

Partition function:

$$Z = \int \mathcal{D}\phi \exp\{ i \int d^4x \, \mathcal{L} \} \quad ; \quad d^4x \to i \underbrace{d\beta}_{\text{sum}} d^3x$$

$\hookrightarrow$ Stat. Mech.

$\mathcal{L}(x) \to \mathcal{L}(t \to i\beta, \vec{x})$

Example:

Pressure $P = \frac{T}{V} \ln Z$

$\boxed{\text{More in Fys 9180 !}}$