# ~~Result~~ "Fasit" Mid Term Exam Fys5120
## Spring semester 2013

Problem 1)


$r = p + k$

Electron self energy

$$-i\Sigma(p) = e_0^2\,\hat{\Sigma}(p)$$
$$= e_0^2\left\{\left[\hat{\Sigma}(p)\right]_{\eta=0} + \eta\left[\Delta\hat{\Sigma}(p)\right]\right\}$$

where $\eta$ is the gauge parameter in the photon propagator $D^{\mu\nu}(k)$ in eq. (1) in Mid-Term problems. Thus $\left[\hat{\Sigma}(p)\right]_{\eta=0}$ correspond to the $(-g^{\mu\nu})$ term of $D^{\mu\nu}(k)$.

Using standard methods in Peskin & Schroeder we find (in $d = 4 - \varepsilon$ dimensions):

$$\left[\hat{\Sigma}(p)\right]_{\eta=0} = 2(d-1)\,B\!\left(\tfrac{d}{2},\tfrac{d}{2}\right)\gamma\cdot p\,(-p^2)^{\left(\frac{d}{2}-2\right)}\frac{i}{(4\pi)^{d/2}}\cdot\frac{\Gamma\!\left(2-\frac{d}{2}\right)}{\Gamma(2)},$$

where $B(\alpha,\beta)$ is defined in eq. (2) of MT-problems.

Note that we have used

$$\underline{(2-d)\,B\!\left(\tfrac{d}{2}-1,\tfrac{d}{2}\right) = -2(d-1)\,B\!\left(\tfrac{d}{2},\tfrac{d}{2}\right)}$$

(Remember $m_0 = 0$)

As usual, we have used the standard integrals

$$I_n \equiv \int \frac{\not{d}l}{[l^2-\Delta+i\varepsilon]^n} \; ; \; \Delta = -x(1-x)p^2$$

$$\not{d}l \equiv \frac{d^d l}{(2\pi)^d} \quad : \qquad (n = 1,2,3,\cdots)$$

Note that $\Delta \cdot I_3 = \left(\frac{d}{4}-1\right)\cdot I_2$ ,

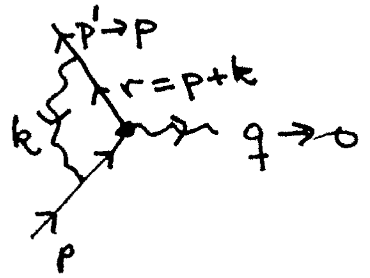
and $B\left(\frac{d}{2}+1, \frac{d}{2}-1\right) = \frac{d}{d-2}\, B\left(\frac{d}{2},\frac{d}{2}\right)$.

Using such relations, we obtain

$$[\Delta\hat{\Sigma}(p)] = -[\hat{\Sigma}(p)]_{\eta=0} \quad \text{, such that}$$

$$\Sigma'(p) = 0 \quad \text{for } \eta = 1 \quad (\text{Landau gauge})$$

Problem 2)




Contribution from vertex correction diagram:

$$\Delta\Gamma^\mu(p'=p;p) = e_0^3\left[\hat{\Gamma}^\mu_{\eta=0} + \eta\cdot\delta\hat{\Gamma}^\mu\right]$$

where $\hat{\Gamma}^\mu_{\eta=0}$ is due to the $(-g^{\mu\nu})$ part of $D^{\mu\nu}(k)$.

We use the following Dirac algebra in $d$ dimensions:

$$\gamma^\alpha \gamma^\mu \gamma_\alpha = (2-d)\gamma^\mu \;;\; \gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu} - (4-d)\gamma^\mu\gamma^\nu$$

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\sigma \gamma_\alpha = -2\gamma^\sigma\gamma^\nu\gamma^\mu + (4-d)\gamma^\mu\gamma^\nu\gamma^\sigma$$

Using also relations for $B(\alpha,\beta)$ functions above, we obtain

$$\hat{\Gamma}^\mu_{\eta=0} = -2(d-1)B\left(\frac{d}{2},\frac{d}{2}\right)\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left\{(-p^2)^{\left(\frac{d}{2}-2\right)}\gamma^\mu + (4-d)(-p^2)^{\frac{d}{2}-3}\gamma p\, p^\mu\right\}$$

Using in addition $\Delta \cdot I_4 = \left(\frac{d}{6}-1\right)\cdot I_3$, we obtain after some tedious calculation that

$$\delta\hat{\Gamma}^\mu = -\left(\hat{\Gamma}^\mu_{\eta=0}\right), \text{ such that}$$

$$\left[\Delta\Gamma^\mu(p',p)\right]_{p'=p} = 0 \quad \text{for } \eta=1 \text{ (Landau gauge)}$$

Problem 3)

Using

$$\frac{\partial}{\partial p_\mu}\left[\gamma\cdot p\,(-p^2)^{\frac{d}{2}-2}\right] =$$

$$\gamma^\mu(-p^2)^{\frac{d}{2}-2} + \gamma\cdot p\cdot\left(\frac{d}{2}-2\right)(-p^2)^{\frac{d}{2}-3}\cdot(-2p^\mu),$$

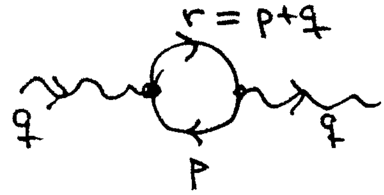
we easily get

$$e_0\,\frac{\partial\left[\Sigma(p)_{\eta=0}\right]}{\partial p_\mu} = -e_0^3\hat{\Gamma}^\mu_{\eta=0}, \text{ and sim. for}$$

"$\eta$-part" : $\quad e_0\frac{\partial}{\partial p_\mu}\Delta\Sigma(p) = -e_0^3\delta\Gamma^\mu$

Problem 4






We obtain (using standard methods)

$$i\,\Pi^{\mu\nu}(q) = -e_0^2\,\hat{\Pi}^{\mu\nu}(q) \quad , \text{ where}$$

$$\hat{\Pi}^{\mu\nu} = f(d)\int_0^1 dx \left\{ \frac{(2-d)}{d}\, g^{\mu\nu}\left[ I_1 + \Delta\cdot I_2 \right] \right.$$
$$\left. - x(1-x)\, I_2\, (2q^\mu q^\nu - q^2 g^{\mu\nu}) \right\},$$

$$I_n = \frac{i(-1)^n}{(4\pi)^{d/2}}\,\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\,\frac{1}{\Delta^{n-d/2}} \;;\; \Delta \equiv -x(1-x)q^2$$

$$f(d) \equiv \mathrm{Tr}(\mathbb{1}_{\text{Dirac}}) = 4 + \mathcal{O}((4-d))$$

We use $\quad I_1 + \Delta\cdot I_2 = \dfrac{d}{d-2}\cdot\Delta\cdot I_2 ,$

giving

$$i\,\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)(-2e_0^2 f(d))\, B(\tfrac{d}{2},\tfrac{d}{2})\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}(-q^2)^{\frac{d}{2}-2}$$

$$\boxed{\equiv +ie_0^2\,(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\hat{\Pi}(q^2)}$$

Renormalized charge $e_R$ given by

$$e_R^2 = e_0^2\left[1 + e_0^2\,\hat{\Pi}(q^2 = -\mu^2)\right],$$

or

$$\alpha_R = \frac{e_R^2}{4\pi} = \alpha(\mu^2) = \alpha_0\left[1 + \alpha_0\cdot 4\pi\,\hat{\Pi}(q^2=-\mu^2)\right]$$

Using
$$\Gamma(2-\tfrac{d}{2})\,\frac{\partial}{\partial\mu}\,(\mu^2)^{\frac{d}{2}-2} = -\Gamma(3-\tfrac{d}{2})\,\frac{2\mu}{(\mu^2)^{3-d/2}},$$
$$= \text{finite !}$$

we find

$$\mu\frac{\partial}{\partial\mu}\alpha(\mu^2) = K\cdot[\alpha(\mu^2)]^2 + \mathcal{O}(\alpha^3)$$

(We had a misprint in eq. (6) of problems)

with $K = \frac{2}{3\pi}$

Solution:

$$\frac{1}{\alpha_2} = \frac{1}{\alpha_1} - K\ln\frac{\mu_2}{\mu_1}$$

Or: for $\mu_1 = \mu$ and $\mu_2 = Q$

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1-\frac{\alpha(\mu^2)}{3\pi}\ln\left(\frac{Q^2}{\mu^2}\right)}$$

I have given max 10 points per problem (- i.e. Max score = 50 points) when correction/reading the written answers