Problem sheet 2

FYS5120-Advanced Quantum Field Theory

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As we discussed in *Problem sheet 1*, dimensional regularization is the most important and most used regularization scheme. The key observation is that an integral such as

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2+i\epsilon)^2}\,, \tag{1}$$

is divergent only if $d \geq 4$. However, if $d < 4$, it will converge. If it is convergent we can Wick rotate, and the answer comes from analytically continuing to d-dimensions.

Problem 2.1: Vacuum Polarization in Scalar Field Theory

Let us consider the interacting theory of two scalar fields ϕ and θ, with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}M^2\theta^2 - g\phi^2\theta\,, \tag{2}$$

defined in $d = 4-2\epsilon$[1] dimensions, where ϵ is the regulator in dimensional regularization.

***a*)** Use dimensional analysis and find the dimensions of the fields and couplings in the Lagrangian. As you later will learn, we generally want couplings to be dimensionless. Introduce a scale μ such that g is dimensionless and write down the modified interaction term.

***b*)** Use the new interaction term you found in the previous exercise and find the one-loop self-energy, $i\Pi(p^2)$, of the θ particle. After introducing Feynman parameters and integrating over the momentum you should find

$$i\Pi(p^2) = -\frac{ig^2}{8\pi^2}\left[\frac{1}{\bar{\epsilon}} + \ln\frac{\mu^2}{m^2} - \int_0^1 dx\,\ln\left(1+\frac{p^2}{m^2}x(x-1)-i\epsilon\right)\right], \tag{3}$$

where we define

$$\frac{1}{\bar{\epsilon}} \equiv \frac{1}{\epsilon} - \gamma_E + \ln 4\pi\,. \tag{4}$$

This might seem as a strange definition, but after we talk about renormalization it will make more sense.

***c*)** The final integration over Feynman parameters are often not given explicit in textbooks, so you are fortunate to get the chance to do it here[2]. We have the integral

$$I = \frac{g^2}{8\pi^2}\int_0^1 dx\,\ln\left(1+\frac{p^2}{m^2}x(x-1)-i\epsilon\right). \tag{5}$$

[1] Don't confuse this ϵ with the one used to shift the propagators $i\epsilon$. It is confusing, but you just have to live with it.
[2] This is a one time thing only as it is quite a mess, but it is useful to have done it.

1. Show that for $p^2 > 4m^2$, we have

$$I = -i\frac{g^2}{8\pi}\sqrt{1-\beta} - \frac{g^2}{4\pi^2}\left[1 + \frac{1}{2}\sqrt{1-\beta}\ln\frac{1-\sqrt{1-\beta}}{1+\sqrt{1-\beta}}\right]. \tag{6}$$

2. Show that for $0 < p^2 < 4m^2$, we have

$$I = \frac{g^2}{4\pi^2}\left[-1 + \sqrt{\beta - 1}\arcsin\sqrt{\beta^{-1}}\right], \tag{7}$$

where

$$\beta = \frac{4m^2}{p^2}. \tag{8}$$

Problem 2.2: One-loop ϕ^3-theory in six dimensions

Consider the scalar ϕ^3-theory action

$$S = \int d^6x\Big(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3\Big). \tag{9}$$

Using dimensional analysis you should find that $[\phi] = m^2$ and $[g] = m^0$. However, if we by analytic continuation change to $d = 6 - 2\epsilon$, g will become dimensionfull. Hence, we introduce $\mu^{(6-d)/2}$ such that g is dimensionless. Now, the scalar self energy diagram is very much like the one you calculated in the previous exercise, so it is not necessary to calculate that again here. Instead, we will calculate the diagram in Fig. 1.

a) First, we write the full one-loop vertex function $\Gamma(p_1, p_2)$ as

$$\Gamma(p_1, p_2) = g(1 + \Lambda(p_1, p_2)). \tag{10}$$

Show that

$$\Lambda(p_1, p_2) = \frac{g^2\mu^{6-d}}{(4\pi)^{d/2}}\Gamma\Big(3 - \frac{d}{2}\Big)\int_0^1 dx\int_0^{1-x} dy\, \Delta^{d/2-3} \tag{11}$$

where

$$\Delta = m^2 - x(1-x)p_1^2 - y(1-y)p_2^2 - 2xy\, p_1\cdot p_2 \tag{12}$$

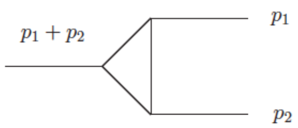

Figure 1: One-loop vertex correction in ϕ^3 theory.

b) Finally, expand the result and show that the full one-loop vertex function takes the from

$$\Gamma(p_1, p_2) = g\left[1 + \frac{g^2}{(4\pi)^3}\left(\frac{1}{2\bar{\epsilon}} + \ln\frac{\mu^2}{m^2} - \int_0^1 dx\int_0^{1-x} dy\, \ln X\right)\right] \tag{13}$$

where we defines $X = \Delta/m^2$.