Problem sheet 3

FYS5120-Advanced Quantum Field Theory

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Problem 3.1: Dirac matrices in d-dimensions

As we are moving from four dimensional space to d-dimensions, contractions and traces with Dirac matrices will change accordingly. Use the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} \equiv 2g^{\mu\nu}\,, \tag{1}$$

and show that we have the following identities

$$\gamma^\mu\gamma_\mu = d \tag{2}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu \tag{3}$$

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\nu\lambda} + (d-4)\gamma^\nu\gamma^\lambda \tag{4}$$

$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_\mu = (d-4)\gamma^\nu\gamma^\lambda\gamma^\rho - 2\gamma^\rho\gamma^\lambda\gamma^\nu\,. \tag{5}$$

Problem 3.2: One-loop vacuum polarization in QED

In massive QED, we have the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi\,, \tag{6}$$

where $F_{\mu\nu}$ are the usual Abelian field strength tensor and D_μ is the covariant derivative acting on fields in the fundamental representation of the gauge group $U(1)$.

a) Show that the one-loop vacuum polarization, $\mathrm{i}\Pi(p^2)$, can be written as.

$$i\Pi^{\mu\nu}(p^2) = (-)e^2\mu^{4-d}\,\mathrm{tr}[\mathbf{1}]\int\frac{d^dk}{(2\pi)^d}\frac{N^{\mu\nu}}{(k^2-m^2+i\epsilon)((k+p)^2-m^2+i\epsilon)}\,, \tag{7}$$

where

$$N^{\mu\nu} = 2k^{\mu\nu} + p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}(k^2 + k\cdot p - m^2)\,. \tag{8}$$

Can you explain where the overall minus sign comes from? Also, can you explain where the factor μ^{4-d} come from?

b) Show that, by symmetry, one has

$$\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu}{(k^2-\Delta)^n} = 0 \tag{9}$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2-\Delta)^n} = \frac{1}{d}\int\frac{d^dk}{(2\pi)^d}\frac{k^2 g^{\mu\nu}}{(k^2-\Delta)^n} \tag{10}$$

c) Use the results from ***b)*** and Feynman parametrization to show that after momentum integration, we have

$$i\Pi^{\mu\nu}(p^2) = i(p^\mu p^\nu - p^2 g^{\mu\nu})\Pi(p^2)\,, \tag{11}$$

where

$$\Pi(p^2) = \frac{2e^2\mu^{4-d}}{(4\pi)^{d/2}}\mathrm{tr}[\mathbf{1}]\Gamma\Big(2-\frac{d}{2}\Big)\int_0^1 dx\, x(1-x)\Delta^{d/2-2}\,. \tag{12}$$

Show that this result is consistent with gauge invariance. *Hint: What is the diagrammatic version of gauge invariance?*

c) Finally, expand around $d = 4 - 2\epsilon$ and show that the vacuum polarization has a single pole in 4 dimensions, i.e.

$$\Pi(p^2)\Big|_{\mathrm{UV}} = \frac{e^2}{12\pi^2}\frac{1}{\epsilon} \tag{13}$$