Problem sheet 5

FYS5120-Advanced Quantum Field Theory

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Problem 5.2: Renormalized perturbation theory

In this exercise you will renormalize QED to one-loop order. To this end, start with the *bare* QED Lagrangian[1]

$$\mathcal{L} = -\frac{1}{4}F_{0\,\mu\nu}F_0^{\mu\nu} + \bar{\psi}_0\Big(i\not{\partial} - e_0\not{A}_0 - m_0\Big)\psi_0,\,. \tag{1}$$

By realizing that the *bare* Lagrangian does not describe the physical parameters, we can rescale the fields and coupling in the following way

$$\psi_0 = \sqrt{\mathcal{Z}_2}\,\psi \tag{2}$$

$$A_{0\,\mu} = \sqrt{\mathcal{Z}_3}\,A_\mu \tag{3}$$

$$m_0 = \mathcal{Z}_m m \tag{4}$$

$$e_0 = \mathcal{Z}_e e\,. \tag{5}$$

The idea from here is that we want to expand around some tree level values for these parameters, which naturally takes the from

$$\mathcal{Z}_i = 1 + \delta_i\,, \tag{6}$$

where δ_i starts at $\mathcal{O}(e^2)$. First, define

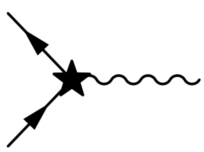
$$\mathcal{Z}_1 = \mathcal{Z}_e\mathcal{Z}_2\sqrt{\mathcal{Z}_3}\,, \tag{7}$$

and show that

$$\delta_e = \delta_1 - \delta_2 - \frac{1}{2}\delta_3 + \mathcal{O}(e^4)\,. \tag{8}$$

Finally, show that the Feynman rules for the counterterms can be written as

$$= -i\delta_3\Big(p^2 g^{\mu\nu} - p^\mu p^\nu\Big)$$



$$= -ie\gamma^\mu\delta_1$$

$$= i\Big(\not{p}\delta_2 - (\delta_2 + \delta_m)m\Big)$$

[1] We ignore the gauge fixing term here.

Problem 5.2: Electron self energy

By using the Feynman rules for counterterms, show that the renormalized two point function can be written as (momentum space)

$$i\mathcal{G}(p) = \frac{i}{\not{p} - m + \Sigma(\not{p})} \tag{9}$$

where

$$\Sigma(\not{p}) = \Sigma_2(\not{p}) - (\delta_m + \delta_2)m + \mathcal{O}(e^4)\,. \tag{10}$$

Calculate the one-loop electron self energy, i.e. $\Sigma_2(\not{p})$, and show that[2]

$$\delta_2 = -\frac{\alpha}{4\pi}\frac{1}{\bar{\epsilon}} \tag{11}$$

$$\delta_m = -\frac{3\alpha}{4\pi}\frac{1}{\bar{\epsilon}}\,, \tag{12}$$

in the $\overline{MS}$ scheme. With these counterterms, you should find the UV-finite expression

$$\Sigma(\not{p}) = \frac{\alpha}{4\pi}\left(2m - \not{p} + 2\int_0^1 dx\Big[(1-x)\not{p} - 2m\Big]\ln\frac{\mu^2}{\Delta}\right) + \mathcal{O}(e^4) \tag{13}$$

where

$$\Delta = xm^2 - x(1-x)p^2\,. \tag{14}$$

Now, there is an IR-divergence in this expression, but we will not worry about these now. The easy way to cure this divergence is to add a photon mass to the photon propagator, but these IR-divergences will be treated more correctly in chapter 20.

Problem 5.3: Vertex correction

In *Problem sheet 4* you found that the vertex correction could be written as

$$\Gamma^\mu_{\text{1-loop}} = F_1^{\text{1-loop}}(q^2)\gamma^\mu + \frac{i}{m}q_\nu S^{\mu\nu}F_2^{\text{1-loop}}(q^2) \tag{15}$$

where the one-loop form factors took the form

$$F_1^{\text{1-loop}} = -2ie^2\mu^{4-d}\int_0^1 dxdydz\delta(x+y+z-1)\int\frac{d^dk}{(2\pi)^d}\frac{\mathcal{N}_1}{(k^2-\Delta+i\epsilon)^3} \tag{16}$$

$$F_2^{\text{1-loop}} = 2ie^2\mu^{4-d}\int_0^1 dxdydz\delta(x+y+z-1)\int\frac{d^dk}{(2\pi)^d}\frac{\mathcal{N}_2}{(k^2-\Delta+i\epsilon)^3} \tag{17}$$

where

$$\Delta = (1-z)^2m^2 - xyq^2 \tag{18}$$

and

$$\mathcal{N}_1 = \frac{(d-2)^2}{d}k^2 - (d-2)\Big((1-z)^2m^2 + xyq^2\Big) + 2z(2m^2 - q^2) \tag{19}$$

$$\mathcal{N}_2 = 2m^2(2z(1-z) + (4-d)(1-z)^2) \tag{20}$$

[2]Remember that $\bar{\epsilon}$ includes $\ln 4\pi$ and γ_E.

By calculating $F_2^{\text{1-loop}}(0)$ you found the correction to the magnetic moment of the electron. In this exercise you will calculate $F_1^{\text{1-loop}}(0)$ and find the counterterm δ_1. First, to leading order we have that

$$F_1(p^2) = 1\,. \tag{21}$$

Hence, it is natural to define the renormalization condition (why?)

$$\Gamma^\mu(0) = \gamma^\mu\,. \tag{22}$$

Use renormalized perturbation theory and show that

$$\delta_1 = -\frac{\alpha}{4\pi}\frac{1}{\bar{\epsilon}}\,, \tag{23}$$

in the $\overline{MS}$ scheme. The last counterterm we need to remove all divergences to one-loop in QED, is δ_3. From the renormalized Lagrangian, you should conclude that δ_3 cancels the divergence in the photon propagator. The one-loop correction to the photon propagator was calculated in *Problem sheet 3*, and you should find that

$$\delta_3 = -\frac{\alpha}{3\pi}\frac{1}{\bar{\epsilon}}\,. \tag{24}$$