Solutions sheet 2

FYS5120-Advanced Quantum Field Theory

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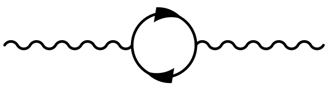
Solution Problem 3.2

a) At one-loop level, one single diagram with a closed fermion loop contributes to $i\Pi(p^2)$. Applying the QED Feynman rules (using $e\mu^{(4-d)/2}$ instead of just e) we write this as

$i\Pi(p^2) =$  $= (-)e^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{tr}\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{p}+m)\right]}{(k^2-m^2+i\epsilon)\left((k+p)^2-m^2+i\epsilon\right)}\,.$

First, we calculate the trace

$$\mathrm{tr}\left[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{p}+m)\right] = \mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\right] + \mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{p}\right] + m^2\mathrm{tr}[\gamma^\mu\gamma^\nu]\,. \tag{1}$$

which can be simplified by using

$$\mathrm{tr}[\gamma^\mu\gamma^\nu] = \mathrm{tr}[\mathbf{1}]g^{\mu\nu} \tag{2}$$
$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = \mathrm{tr}[\mathbf{1}]\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\,, \tag{3}$$

giving the numerator

$$N^{\mu\nu} = 2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}\left(k^2 + k\cdot p - m^2\right)\,. \tag{4}$$

Hence, the vacuum polarization tensor takes the form

$$i\Pi^{\mu\nu}(p^2) = (-)e^2\mu^{4-d}\,\mathrm{tr}[\mathbf{1}]\int\frac{d^dk}{(2\pi)^d}\frac{N^{\mu\nu}}{(k^2-m^2+i\epsilon)((k+p)^2-m^2+i\epsilon)}\,. \tag{5}$$

Closed fermion loops will always give an overall minus sign and the trace of product of Dirac matrices. The reason is that you have to commute the fermion operators in such a way that you can Wick contract to get propagators. Since fermion operators anti-commute, this will give an overall minus sign.

b) The rank-1 tensor integrals vanish by parity, since it corresponds to the integral over a symmetric domain of an odd function. For a similar reason, the rank-2 integral vanishes unless $\mu = \nu$. Hence, it must be proportional to the metric tensor $g^{\mu\nu}$, giving

$$\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2-\Delta)^n} = A(\Delta)\,g^{\mu\nu}\,. \tag{6}$$

In order to find $A(\Delta)$ we can simply contract both sides with $g_{\mu\nu}$, giving

$$A(\Delta) = \frac{1}{d}\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-\Delta)^n} \tag{7}$$

showing that

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2-\Delta)^n} = \frac{1}{d}\int \frac{d^d k}{(2\pi)^d} \frac{k^2 g^{\mu\nu}}{(k^2-\Delta)^n} \tag{8}$$

***c*)** Now, let us use Feynman parametrization to combine the denominators

$$\begin{aligned}\frac{1}{(k^2-m^2+i\epsilon)((k+p)^2-m^2+i\epsilon)} &= \int_0^1 dx\, \frac{1}{(k^2+2x\,k\cdot p+xp^2-m^2)^2} \\ &= \int_0^1 dx\, \frac{1}{(k^2+x(1-x)p^2-m^2+i\epsilon)^2}\end{aligned} \tag{9}$$

where we in the last step shifted $k^\mu \to k^\mu + xp^\mu$. However, this shift must also be made in the numerator, giving

$$N^{\mu\nu} = 2k^\mu k^\nu - 2x(1-x)p^\mu p^\nu - g^{\mu\nu}(k^2 - x(1-x)p^2 - m^2) + (\text{linear terms in } k)\,. \tag{10}$$

In the previous exercise we argued that linear terms would vanish after integration, so these can be dropped. Also, the $k^\mu k^\nu$ term can be replaced according to Eq. (8), reducing the numerator to

$$N^{\mu\nu} = -k^2\left(1-\frac{2}{d}\right)g^{\mu\nu} - 2x(1-x)p^\mu p^\nu + g^{\mu\nu}(x(1-x)p^2+m^2)$$

Summarizing, we have found that the vacuum polarization have the form

$$i\Pi^{\mu\nu}(p^2) = e^2\mu^{4-d}\mathrm{tr}[\mathbf{1}]\int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{k^2\left(1-\frac{2}{d}\right)g^{\mu\nu} + 2x(1-x)p^\mu p^\nu - g^{\mu\nu}(x(1-x)p^2+m^2)}{(k^2-\Delta+i\epsilon)^2} \tag{11}$$

where $\Delta = m^2 - x(1-x)p^2$. Let us do the integration term by term using the integrals we derived in *problem sheet 1*

***Term* 1 :**

$$\begin{aligned}\left(1-\frac{2}{d}\right)g^{\mu\nu}\int \frac{d^d k}{(2\pi)^2}\frac{k^2}{(k^2-\Delta+i\epsilon)^2} &= g^{\mu\nu}\left(1-\frac{d}{2}\right)\frac{i}{(4\pi)^{d/2}}\Gamma\left[1-\frac{d}{2}\right]\Delta^{d/2-1} && (12) \\ &= g^{\mu\nu}\frac{i}{(4\pi)^{d/2}}\Gamma\left[2-\frac{d}{2}\right]\Delta^{d/2-1} && (13)\end{aligned}$$

where we used that

$$\Gamma\left[2-\frac{d}{2}\right] = \left(1-\frac{d}{2}\right)\Gamma\left[1-\frac{d}{2}\right]. \tag{14}$$

***Term* 2, 3 :** These terms does not have any dependence on k in the numerator, so we can use

$$\int \frac{d^d k}{(2\pi)^{d/2}}\frac{1}{(k^2-\Delta+i\epsilon)^2} = \frac{i}{(4\pi)^{d/2}}\Gamma\left[2-\frac{d}{2}\right]\Delta^{d/2-2} \tag{15}$$

Inserting these results, we find

$$\begin{aligned}i\Pi^{\mu\nu}(p^2) =& e^2\mu^{4-d}\mathrm{tr}[\mathbf{1}]\frac{i}{(4\pi)^{d/2}}\Gamma\left[2-\frac{d}{2}\right] && (16) \\ &\times \int_0^1 dx\left(g^{\mu\nu}\Delta + 2x(1-x)p^\mu p^\nu - g^{\mu\nu}(x(1-x)p^2+m^2)\right)\Delta^{d/2-2} && (17)\end{aligned}$$

and if we use that $\Delta - m^2 = -x(1-x)p^2$, we finally find

$$i\Pi^{\mu\nu}(p^2) = 2ie^2\mu^{4-d}\text{tr}[\mathbf{1}]\frac{i}{(4\pi)^{d/2}}\Gamma\Big[2-\frac{d}{2}\Big](p^\mu p^\nu - p^2 g^{\mu\nu})\int_0^1 dx\, x(1-x)\Delta^{d/2-2} \tag{18}$$

If we now define

$$\Pi(p^2) = \frac{2e^2\mu^{4-d}}{(4\pi)^{d/2}}\text{tr}[\mathbf{1}]\Gamma\Big(2-\frac{d}{2}\Big)\int_0^1 dx\, x(1-x)\Delta^{d/2-2}\,, \tag{19}$$

we have shown that the vacuum polarization tensor takes the form

$$\Pi^{\mu\nu}(p^2) = (p^\mu p^\nu - p^2 g^{\mu\nu})\Pi(p^2)\,. \tag{20}$$

This structure could actually have been guessed from gauge invariance. The diagrammatic manifestation of gauge invariance is known as the Ward identity, which says that

$$p_\mu \Pi^{\mu\nu} = 0\,, \tag{21}$$

and we can immediately see that the form we have found satisfies this identity. From the discussion session it became clear that the link between the Ward identity and Gauge invariance was not obvious. So, let me try to make the link a little more clear.

Ward Identity

The defining equation for the vector potential $A_\mu(x)$ without choosing a gauge is

$$\partial_\nu \partial^\nu A_\mu(x) = 0\,, \tag{22}$$

This equation is solved by the ansatz,

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\varepsilon(k)\, e^{-ik\cdot x}\,. \tag{23}$$

We know that in a gauge invariant Abelian theory, the vector potential transform as

$$A_\mu^{'} = A_\mu + \partial_\mu \alpha\,. \tag{24}$$

If we insert for the solution in Eq. (23) and use the Fourier transform

$$\alpha(x) = \int \frac{d^4k}{(2\pi)^4}\tilde{\alpha}(k)\, e^{-ik\cdot x} \tag{25}$$

we find

$$\int \frac{d^4k}{(2\pi)^4}\varepsilon_\mu^{'}(k)\, e^{-ik\cdot x} = \int \frac{d^4k}{(2\pi)^4}\varepsilon_\mu(k)\, e^{-ik\cdot x} + \int \frac{d^4k}{(2\pi)^4}(-ik_\nu)\tilde{\alpha}(k)\, e^{-ik\cdot x}\,, \tag{26}$$

giving that the polarization tensor transform as

$$\varepsilon_\mu^{'}(k) = \varepsilon_\mu(k) - ik_\mu\tilde{\alpha}(k)\,. \tag{27}$$

Let us then consider an arbitrary QED process involving an external photon with momentum k. We can write the amplitude for such a process as

$$\mathcal{M}(k) = \varepsilon_\mu(k)\mathcal{M}^\mu(k)\,. \tag{28}$$

Since ε_μ always appear in such amplitudes, we have extracted it and defined $\mathcal{M}^\mu(k)$ to be the rest of the amplitude (which in QED is a product of spinors and Dirac matrices). According to Eq. (27), the amplitude transform under gauge transformation as

$$\mathcal{M}(k) \to \mathcal{M}^{'}(k) = \varepsilon_\mu(k)\mathcal{M}^\mu(k) - i\tilde{\alpha}(k)k_\mu\mathcal{M}^\mu(k))\,. \tag{29}$$

If the amplitude is to be gauge invariant, we must have the follwing identity

$$k_\mu \mathcal{M}^\mu(k)) = 0\,, \tag{30}$$

which is known as the *Ward identity.* It is essentially a statement of current conservation, which is a consequence of the gauge symmetry in QED. This claim is not obvious from the above heuristic derivation. For a formal proof of this claim we would need to go at the level of correlation functions, and find what is known as the Ward-Takahashi identities. These are independent of perturbation theory and are essential in the proof of renormalization in QED. Hence, we will talk more about them when we get to that point.

c) To find the UV-divergent part of the vacuum polarization, we simply expand all terms in $d = 4 - 2\epsilon$, where it is understood that $\epsilon \to 0$. We use the following expressions

$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{31}$$

$$A^\epsilon = 1 + \epsilon \ln A + \mathcal{O}(\epsilon^2) \tag{32}$$

giving (we can now use that $\mathrm{tr}[\mathbf{1}] = 4$ in four dimensions)

$$\begin{aligned}\Pi(p^2) &= \frac{8e^2}{(4\pi)^2}\Big(\frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\Big)\Big(1 + \epsilon \ln \mu^2 + \mathcal{O}(\epsilon^2)\Big))\Big(1 + \epsilon \ln 4\pi + \mathcal{O}(\epsilon^2)\Big) \\ &\quad \times \int_0^1 dx\, x(1-x)\Big(1 - \epsilon \ln \Delta + \mathcal{O}(\epsilon^2)\Big) \\ &= \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\Big(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{\Delta} + \mathcal{O}(\epsilon)\Big)\,. \end{aligned} \tag{33}$$

It follows that the UV-divergent part is given by

$$\Pi(p^2)\Big|_{\mathrm{UV}} = \frac{e^2}{2\pi^2}\frac{1}{\epsilon}\int_0^1 dx\, x(1-x) = \frac{e^2}{12\pi^2}\frac{1}{\epsilon} \tag{34}$$