Problem sheet 1

FYS5120-Advanced Quantum Field Theory

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In classical physics, the vacuum was considered to be free empty space with no physical properties of its own. However, according to quantum field theory a vacuum contains particles, the numbers of which are in a continuous state of fluctuation and can be thought of as popping in and out of existence.

The first encounter of problems with the vacuum energy that you have faced is probably through the procedure of canonical quantization of scalar fields. For a scalar field ϕ the canonical quantization procedure will give a Hamiltonian of the form

$$\begin{aligned} H &= \int \frac{d^3k}{(2\pi)^3}\, \omega_k \Big(a_k^\dagger a_k + \frac{1}{2}[a_k, a_k^\dagger] \Big) \\ &= \int \frac{d^3k}{(2\pi)^3}\, \omega_k \Big(a_k^\dagger a_k + \frac{1}{2}(2\pi)^3 \delta^{(3)}(0) \Big) \\ &= \int \frac{d^3k}{(2\pi)^3}\, \omega_k \Big(a_k^\dagger a_k + \frac{1}{2} V \Big)\,, \end{aligned} \tag{1}$$

where the volume V is defined as

$$V = (2\pi)^3 \delta^{(3)}(0) = \int d^3x\,. \tag{2}$$

This implies that the vacuum has an infinite energy

$$E_0 = \langle 0|H|0\rangle = \frac{V}{2} \int \frac{d^3k}{(2\pi)^3} \omega_k = \infty\,. \tag{3}$$

As you may have learned, the way out of this problem is to use that the energy of the vacuum is not measurable. Only energy diffenrences are measurable, and in these differences the *zero-point energy* drops out. Mathematically this is fixed by *normal ordering* the Hamiltonian, giving

$$H = \int \frac{d^3k}{(2\pi)^3}\, \omega_k\, a_k^\dagger a_k\,, \tag{4}$$

which give $\langle 0|H|0\rangle = 0$. However, more attention is required whenever the mode spectrum and subsequently the vacuum energy is modified by boundary conditions. This is what we will tackle in this problem sheet.

Problem 1.1: Casimir effect

The Lagrangian for a free electromagnetic field is given by

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \tag{5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu$ for Abelian gauge fields, giving the source free Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0\,. \tag{6}$$

In Coulomb gauge ($A^0 = 0$ and $\nabla \cdot \mathbf{A} = 0$) the most general solution is (Do not confuse the argument x in the equation below with the x-coordinate as this is a four vector),

$$\mathbf{A}(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{\lambda=1}^{2}\Big(a_\lambda(\boldsymbol{k})\boldsymbol{\epsilon}_\lambda(\boldsymbol{k})e^{-ik\cdot x} + a_\lambda^\dagger(\boldsymbol{k})\boldsymbol{\epsilon}_\lambda(\boldsymbol{k})e^{ik\cdot x}\Big)\,, \tag{7}$$

where $w_k = |\mathbf{k}|$ and $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ are the transverse polarization vectors.

We will now consider the electromagnetic field in the space between two infinitely large, parallel, uncharged, perfectly conducting plates located at $z = 0$ and $z = a$ (each plate is in the xy plane). In order to have the appropriate boundary conditions for the electromagnetic fields, the components of the wave-vectors corresponding to their Fourier transforms should satisfy

$$k_i = \frac{n_i \pi}{x_i}\;, \qquad i = 1, 2, 3\;, \tag{8}$$

with n_i being non-negative integers and x_i are the Cartesian dimensions of the problem (here we use $x_1 = x_2 = L$ and $x_3 = a$). The Hamiltonian for this system can be shown to be[1],

$$H = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\sum_{n=1}^{\infty}\omega_{k,n}\sum_{\lambda=1}^{2}\Big(a_\lambda^\dagger(k_x,k_y,n)a_\lambda(k_x,k_y,n) + a_\lambda(k_x,k_y,n)a_\lambda^\dagger(k_x,k_y,n) \tag{9}$$

$$+ \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\, w_k\Big(a^\dagger(k_x,k_y)a(k_x,k_y,n) + a(k_x,k_y)a^\dagger(k_x,k_y)\Big) \tag{10}$$

where

$$\omega_{k,n} = \sqrt{k_x^2 + k_y^2 + \Big(\frac{n\pi}{a}\Big)} \tag{11}$$

a) Use the following commutation relations

$$[a_\lambda(k_x,k_y,n), a_{\lambda'}^\dagger(k_x',k_y',n')] = (2\pi)^2\delta_{nn'}\delta_{\lambda\lambda'}\delta(k_x - k_x')\delta(k_y - k_y') \tag{12}$$

$$[a(k_x,k_y), a^\dagger(k_x',k_y')] = (2\pi)^2\delta(k_x - k_x')\delta(k_y - k_y') \tag{13}$$

and show that the vacuum energy can be written as

$$E_0 = \frac{L^2}{2}\int \frac{d^2k}{(2\pi)^2}\left(2\sum_{n=1}^{\infty}\sqrt{k_x^2 + k_y^2 + \Big(\frac{n\pi}{a}\Big)^2} + \sqrt{k_x^2 + k_y^2}\right) \tag{14}$$

b) The expression above is divergent, so we have to regularize it. First introduce a regulator μ and calculate the integral[2]

$$I = \int d^2k \frac{1}{(k^2 + \mu^2)^\alpha} \tag{15}$$

and make the last term in Eq. (14) vanish, i.e. show that we have

$$E_0 = L^2 \int \frac{d^2k}{(2\pi)^2}\sum_{n=1}^{\infty}\sqrt{k_x^2 + k_y^2 + \Big(\frac{n\pi}{a}\Big)^2}\,. \tag{16}$$

[1] The procedure of finding this Hamiltonian is very similar to the calculation you probably did in FYS4170 for scalar fields, with the addition of boundary conditions and polarizations. Hence, it is quite a mess and not the purpose of this exercise.

[2] it is understood that μ should be sent to zero at the end of a calculation.

As you probably know we can only swap the order of summation and integration if the integrand converges. However, it does not look like the above integrand converges. Regulate the integrand with the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\,, \tag{17}$$

and show that

$$\mathscr{E} = \frac{E_0}{L^2} = -\frac{\pi^2}{720a^3}\,. \tag{18}$$

The Casimir force can then be found by calculating

$$F(a) = -\frac{d\mathscr{E}}{da}\,. \tag{19}$$

Problem 1.2: Dimensional Regularization and Feynman Parameters

A typical loop integral has the form

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - m^2 + i\epsilon)^n} \tag{20}$$

Because of the Minkowski signature we use the $i\epsilon$ prescription to shift the poles in propagators. The integral above can be evaluated using Cauchy's contour integral formula, but an easier way is to use what is called *Wick rotation.* The $i\epsilon$ prescription tells us that in the k^0-plane, the pole at $k^0 > 0$ is shifted below the real axis and the pole at $k^0 < 0$ is shifted above the real axis. Therefore we can change the integration path in the complex k^0-plane, by rotating counterclockwise from the real axis to the imaginary axis[3]. The effect of this rotation is that we change from an Minkowskian signature to an Euclidean signature, i.e. we can define the Euclidean four momentum variable $k_E = (ik_E^0, \mathbf{k}_E)$, giving $k_E^2 = -k^2$.

With these manipulations the loop integral takes the form

$$i(-1)^n \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2 + m^2)^n} \tag{21}$$

a) As you will learn the most powerful regularization scheme that preserves all the symmetries in QFT is to use dimensional regularization. Hence, take the integral to d-dimensions and prove the following identities

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{(k_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\Delta^{\frac{d}{2}-n} \tag{22}$$

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{p_E^2}{(k_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\Delta^{1+\frac{d}{2}-n} \tag{23}$$

where you will have use for the Euler β-function

$$\beta(a,b) = \int_0^\infty dx\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,. \tag{24}$$

b) In loop calculations we often end up with denominators that consist of products of different propagators. A very useful method for dealing with these is to introduce *Feynman parameters.*

Prove by induction the formula for the Feynman parametrization of n propagators

$$\frac{1}{A_1^{a_1}\dots A_n^{a_n}} = \frac{\Gamma(a_1+\dots+a_n)}{\Gamma(a_1)\cdots\Gamma(a_1)}\int_0^1 dx_1\dots dx_n\,\frac{\delta(1-x_1+\dots+x_n)\,x_1^{a_1-1}\cdots x_n^{a_n-1}}{(x_1A_1+\dots+x_nA_n)^{a_1+\dots+a_n}}\,, \tag{25}$$

[3]We have to rotate counterclockwise to avoid crossing the poles.