Problem sheet 10: Renormalization of QED part 2

FYS5120-Advanced Quantum Field Theory

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$\rightsquigarrow$ These problems are scheduled for discussion on **Wednesday, 13 April 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

In the last couple of problem sheets you have wrestled with the procedure of integrating out high energy degrees of freedom and showed how we can obtain finite amplitudes by renormalizing the parameters of the theory. While the idea of integrating out momenta up to a cut-off Λ_0 is very intuitive, in more complicated scenarios it becomes very cumbersome to perform. More seriously, in a gauge theory, simply imposing a cut-off does not preserve gauge invariance. That is not to say that momentum cut-off is entirely useless, as it gives a clear indication to the origin of the divergence[1]. We should, however, use a method that preserves all the symmetries of the theory. Probably the most used and arguably most useful is that of dimensional regularization. There are other choices that are valid for certain theories; for example, Pauli-Villars regularization works for Abelian gauge theories, but not for non-Abelian gauge theories. Dimensional regularization preserves all the symmetries, so we will mostly stick with this method. However, the disadvantage is that it is not as transparent in which momentum region the divergence originate[2].

Whether an integral diverges or not is largely determined by power counting, e.g. we have that the integral

$$\int^{\Lambda} \frac{d^4k}{k^4} \sim \ln \Lambda \,, \tag{1}$$

is logarithmically divergent. We observe that if the physical dimension were less than four, this integral would no longer diverge in the UV-region. That suggests we can regulate an integral by computing in some generic number of dimensions d[3]. Since the dimension is generic, the integral is not divergent until we specify the physical dimension. In order to 'approach' the physical dimension, we have to analytically continue the result of our d-dimensional theory through a non-integer value of d. Hence, we introduce a regulator ϵ and define that $d = 4 - 2\epsilon$[4], where ϵ is in general complex. In momentum cut-off, we send the regulator to infinity at the end of a calculation, while in dimensional regularization the divergence will manifest itself for $\epsilon \to 0$.

A we have 'changed' the space-time dimensions, we have to re-evaluate the mass dimension of fields and couplings in the Lagrangian. In order to keep the action dimensionless, we must have

[1] Meaning if the divergence is in the UV or IR region of momentum space.

[2] Should also mention that it is purely a perturbative regulator scheme.

[3] This argument is kind of heuristic, but using the Wilsonian renormalization procedure one can deduce that this is indeed true.

[4] We should not confuse this ϵ with the Feynman prescription $i\epsilon$ of shifting the poles in propagators.

that the Lagrangian has the mass dimension

$$\Big[\int d^d x\mathcal{L}\Big] = m^0 \implies [\mathcal{L}] = m^d\,, \tag{2}$$

i.e. the mass dimension of fields changes as well. E.g. for a scalar field we must have

$$[(\partial_\mu\phi)^2] = m^d \implies [\phi] = m^{1-\epsilon/2}\,, \tag{3}$$

leading to a dimensionful coupling

$$[\lambda] = m^{(4-d)/2}\,. \tag{4}$$

In order to keep the coupling dimensionless, we can replace

$$\lambda \to \mu^{(4-d)/2}\lambda(\mu)\,, \tag{5}$$

where $\lambda(\mu)$ is dimensionless, and μ is some arbitrary energy scale. It is important to stress that μ is not a cut-off; it is merely a scale we introduce, allowing us to use dimensionless couplings. This scale is most often set at the typical scale of the experiment we are interested in explaining and is referred to as the *renormalization scale*. In practice, this means that instead of calculating with the coupling that appears in the Lagrangian, we will replace it with $\lambda\mu^\epsilon$. Just to be clear, observables should not depend on this scale, which is a useful property we will exploit later on.

Problem 19: Dimensional Regularization

A typical loop integral has the form

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - m^2 + i\epsilon)^n} \tag{6}$$

Because of the Minkowski signature we use the $i\epsilon$ prescription to shift the poles in propagators. The integral above can be evaluated using Cauchy's contour integral formula, but an easier way is to use what is called *Wick rotation*. The $i\epsilon$ prescription tells us that in the k^0-plane, the pole at $k^0 > 0$ is shifted below the real axis and the pole at $k^0 < 0$ is shifted above the real axis. Therefore we can change the integration path in the complex k^0-plane, by rotating counterclockwise from the real axis to the imaginary axis[5]. The effect of this rotation is that we change from an Minkowskian signature to an Euclidean signature, i.e. we can define the Euclidean four momentum variable $k_E = (ik_E^0, \mathbf{k}_E)$, giving $k_E^2 = -k^2$.

With these manipulations the loop integral takes the form

$$i(-1)^n \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2 + m^2)^n} \tag{7}$$

Let us see how these integrals behave in d-dimensions.

a) Show that

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{(k_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\Delta^{\frac{d}{2}-n} \tag{8}$$

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{p_E^2}{(k_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\Delta^{1+\frac{d}{2}-n} \tag{9}$$

where you will have use for the Euler β-function

$$\beta(a,b) = \int_0^1 dx\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,. \tag{10}$$

[5] We have to rotate counterclockwise to avoid crossing the poles.

b) As we keep the dimension generic we have to modify the Dirac gamma matrix identities

$$\gamma^\mu \gamma_\mu = d \tag{11}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = (2-d)\gamma^\nu \tag{12}$$
$$\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu = 4g^{\nu\lambda} + (d-4)\gamma^\nu \gamma^\lambda \tag{13}$$
$$\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma_\mu = (4-d)\gamma^\nu \gamma^\lambda \gamma^\rho - 2\gamma^\rho \gamma^\lambda \gamma^\nu\,. \tag{14}$$

We will use these identities later, but unless you really want to derive them it suffices to just use them.

Problem 19: Vacuum Polarization

The process of Wilsonian renormalization showed that in order to render our theories finite we have to tune the parameters of the theory. Following in the same spirit we have that the QED action is tuned to

$$S_0[A_0, \psi_0] \to S[A,\psi] + S^{\mathrm{CT}}[A,\psi] \tag{15}$$

a) Let us derive the form of the renormalized action without integrating out degrees of freedom, but by simply redefining the fields and parameters in the naive bare action one often starts with. Start with the bare action for massive QED

$$S_0[A_0,\psi_0] = \int d^dx \Big(-\frac{1}{4} F_{0\,\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 \big(i\partial\!\!\!/ - e_0 A\!\!\!/_0 - m_0 \big) \psi_0 \Big)\,, \tag{16}$$

and introduce renormalization parameters for each field and parameter in the action to show that

$$S[A,\psi] = \int d^dx \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \big(i\partial\!\!\!/ - eA\!\!\!/ - m \big) \psi \Big) \tag{17}$$

$$S^{\mathrm{CT}}[A,\psi] = \int d^dx \Big(-\frac{1}{4} \delta_3 F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \big(i\delta_2 \partial\!\!\!/ - e\delta_1 A\!\!\!/ - (\delta_m + \delta_2) m \big) \psi \Big)\,, \tag{18}$$

b) Show that the one-loop vacuum polarization, $i\Pi^{\mu\nu}(p^2)$, can be written as.

$$i\Pi^{\mu\nu}(p^2) = (-)e^2 \mu^{4-d}\, \mathrm{tr}[\mathbf{1}] \int \frac{d^dk}{(2\pi)^d} \frac{N^{\mu\nu}}{(k^2 - m^2 + i\epsilon)((k+p)^2 - m^2 + i\epsilon)}\,, \tag{19}$$

where

$$N^{\mu\nu} = 2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}(k^2 + k\cdot p - m^2)\,. \tag{20}$$

Can you explain where the overall minus sign comes from?

c) Show that, by symmetry, one has

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2 - \Delta)^n} = 0 \tag{21}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - \Delta)^n} = \frac{1}{d} \int \frac{d^dk}{(2\pi)^d} \frac{k^2 g^{\mu\nu}}{(k^2 - \Delta)^n} \tag{22}$$

Use the results from *Problem 19* and Feynman parametrization to show that after momentum integration, we have

$$i\Pi^{\mu\nu}(p^2) = i(p^\mu p^\nu - p^2 g^{\mu\nu})\Pi(p^2)\,, \tag{23}$$

where

$$\Pi(p^2) = \frac{2e^2\mu^{4-d}}{(4\pi)^{d/2}}\mathrm{tr}[\mathbf{1}]\Gamma\Big(2-\frac{d}{2}\Big)\int_0^1 dx\, x(1-x)\Delta^{d/2-2}\,. \tag{24}$$

which has the momentum form we expect from the Ward identity.

d) Finally, expand around $d = 4 - 2\epsilon$ and show that the vacuum polarization in QED has a single pole in 4 dimensions, i.e.

$$\Pi(p^2)\Big|_{\mathrm{UV}} = \frac{e^2}{12\pi^2}\frac{1}{\epsilon} \tag{25}$$

Hint: the expansion can be messy for the full expression, so to extract the UV-divergence you can simply set m=0.