Problem sheet 11: Renormalization of QED part 2

FYS5120-Advanced Quantum Field Theory

Lasse Lorentz Braseth, Jonas Eidesen

⇝ These problems are scheduled for discussion on **Wednesday, 20 April 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

In the last couple of problem sheets, we have argued and shown that the naive action leads to mathematically ill-defined quantities. There are several ways of treating this with more care, but the simplest is to say that the theory only makes sense when the following tuning is finite

$$S_0[A_0, \psi_0] \to S[A, \psi] + S^{\mathrm{CT}}[A, \psi] \tag{1}$$

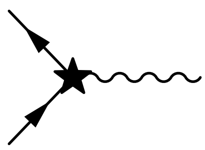
where

$$S[A, \psi] = \int d^d x \bigg(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\Big(i\not{\partial} - e\not{A} - m \Big)\psi \bigg) \tag{2}$$

$$S^{\mathrm{CT}}[A, \psi] = \int d^d x \bigg(-\frac{1}{4} \delta_3 F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\Big(i\delta_2\not{\partial} - e\delta_1\not{A} - (\delta_m + \delta_2) m \Big)\psi \bigg) , \tag{3}$$

It follows that we have to extend the naive Feynman rules to include these counterterms. The terminology for this procedure is *renormalized perturbation theory*, giving the additional rules

$$= -i\delta_3 \Big(p^2 g^{\mu\nu} - p^\mu p^\nu \Big)$$



$$= -ie\gamma^\mu \delta_1$$

$$= i\Big(\not{p}\delta_2 - (\delta_2 + \delta_m) m \Big)$$

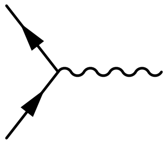
In this problem set, we will continue on the quest of renormalizing QED to one-loop order. However, we will first take a slight detour and actually make a prediction, namely predict the anomalous magnetic moment of the electron. This result[1] is to date the most accurate in all of physics and was (and is) a great success of quantum field theory.

[1] Not the one-loop result, but by including even higher orders.

Problem 20: Anomalous Magnetic Moment of the Electron

The general procedure when looking for radiative corrections to the way spinors interact with photons, such as corrections to $g-2$, is to calculate off-shell matrix elements ('exchange' of a virtual photon). The tree-level matrix element for a electron-electron interaction with a photon is given by (you can think of the photon as a unconstrained external magnetic field)

$i\mathcal{M}_0 =$



$$= -ie\bar{u}(p')\gamma^\mu u(p) \tag{4}$$

where momentum conservation demands that $q^\mu = p'^\mu - p^\mu$.

Now, it is not easy to see how we can find the correction to g from calculating corrections to this matrix element. However, we remember from FYS4170 that scalars and spinors interact differently with photons, and the appearance of the term $F_{\mu\nu}S^{\mu\nu}$ for spinors was this difference. Hence, we should be able to decompose the matrix element into one scalar interaction and one term that takes the form $F_{\mu\nu}S^{\mu\nu}$.

a) For on-shell spinors, show the following identity (particles with same mass)

$$\bar{u}(p')\gamma^\mu u(p) = \frac{(p'^\mu + p^\mu)}{2m}\bar{u}(p')u(p) + \frac{i}{m}\bar{u}(p')q_\nu S^{\mu\nu}u(p)\,, \tag{5}$$

giving the amplitude

$$i\mathcal{M}_0 = -e\frac{(p^\mu + p'^\mu)}{2m}\bar{u}(p')u(p) - i\frac{e}{m}\bar{u}(p')q_\nu S^{\mu\nu}u(p)\,. \tag{6}$$

Can you recognize the structure of this amplitude? Relate it to the way scalars and spinors interact with the electromagnetic field.

(*Hint: To find the identity, use the Dirac equation and that the spinors, u, satisfy plane wave solutions.*)

b) For higher order processes, we can parametrize our ignorance in the following way

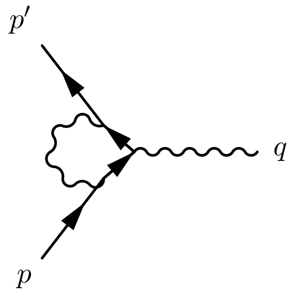
$$i\mathcal{M}^\mu = \bar{u}(p')\Big(-ie\Gamma^\mu(p,p')\Big)u(p) \tag{7}$$

where the vertex correction takes the general form

$$\Gamma^\mu = F_1(q^2)\gamma^\mu + \frac{i}{m}q_\nu S^{\mu\nu}F_2(q^2)\,. \tag{8}$$

Then, calculate

$\Gamma^\mu_{\text{1-loop}}(p,p') =$



where

$$\Gamma^\mu_{\text{1-loop}}(p,p') = -ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{\gamma^\nu(\not{p}' + \not{k} + m)\gamma^\mu(\not{p} + \not{k} + m)\gamma_\nu}{\big((p'+k)^2 - m^2 + i\epsilon\big)\big((p+k)^2 - m^2 + i\epsilon\big)\big(k^2 + i\epsilon\big)} \tag{9}$$

and show that it can be written as

$$\Gamma^\mu_{\text{1-loop}} = F_1^{\text{1-loop}}(q^2)\gamma^\mu + \frac{i}{m} q_\nu S^{\mu\nu} F_2^{\text{1-loop}}(q^2) \tag{10}$$

where the one-loop form factors take the form

$$F_1^{\text{1-loop}} = -2ie^2\mu^{4-d} \int_0^1 dxdydz\delta(x+y+z-1) \int \frac{d^dk}{(2\pi)^d} \frac{\mathcal{N}_1}{(k^2-\Delta+i\epsilon)^3} \tag{11}$$

$$F_2^{\text{1-loop}} = 2ie^2\mu^{4-d} \int_0^1 dxdydz\delta(x+y+z-1) \int \frac{d^dk}{(2\pi)^d} \frac{\mathcal{N}_2}{(k^2-\Delta+i\epsilon)^3} \tag{12}$$

where

$$\Delta = (1-z)^2 m^2 - xyq^2 \tag{13}$$

and

$$\mathcal{N}_1 = \frac{(d-2)^2}{d} k^2 - (d-2)\Big((1-z)^2 m^2 + xyq^2\Big) + 2z(2m^2 - q^2) \tag{14}$$

$$\mathcal{N}_2 = 2m^2\big(2z(1-z) + (4-d)(1-z)^2\big) \tag{15}$$

(*NB! The numerator in Eq. (9) is nasty and takes quite some effort to rewrite. Now, since F_2 is finite in 4-dimensions, you can use the Dirac matrix relations that you are used to. However, in a future exercise, you will calculate F_1 as well, so I recommend you to do it in d-dimensions. Also, remember that Γ^μ is squeezed between $\bar{u}$ and u, so there are several simplifications to make using the Dirac equation.*)

c) Finally, show that the one-loop correction to the anomalous magnetic moment of the electron is given by

$$g - 2 = 2F_2^{\text{1-loop}}(0) = \frac{\alpha}{\pi} \tag{16}$$

This is the prediction made by Schwinger, Feynman and Tomonaga in 1948 which culminated in a Nobel prize in physics in 1965. Together with the calculation of the Lamb shift, this is the result that convinced theoretical physicists at the time that QFT was on the right track in the understanding of elementary particle interactions.

Problem 21: Vertex Correction

By calculating $F_2^{\text{1-loop}}(0)$ we found the correction to the magnetic moment of the electron. In this exercise we will calculate $F_1^{\text{1-loop}}(0)$ and find the counterterm δ_1. First, to leading order we have that

$$F_1(p^2) = 1\,. \tag{17}$$

Hence, it is natural to define the renormalization condition (why?)

$$\Gamma^\mu(0) = \gamma^\mu\,. \tag{18}$$

Use renormalized perturbation theory and show that[2]

$$\delta_1 = -\frac{\alpha}{4\pi}\Big(\frac{1}{\epsilon} + \ln 4\pi - \gamma_E\Big) := -\frac{\alpha}{4\pi}\frac{1}{\bar{\epsilon}}\,, \tag{19}$$

in the $\overline{MS}$ scheme. Further, let us extract the counterterm δ_3. From the renormalized Lagrangian, you should conclude that δ_3 cancels the divergence in the photon propagator. The one-loop correction to the photon propagator was calculated in *Problem sheet 10*, and you should find that

$$\delta_3 = -\frac{\alpha}{3\pi}\frac{1}{\bar{\epsilon}}\,. \tag{20}$$

[2] γ_E is the Euler-Mascheroni constant.

Problem 22: Electron Self Energy

By using the Feynman rules for counterterms, show that the renormalized two point function can be written as

$$i\mathcal{G}(p) = \frac{i}{\not{p} - m + \Sigma(\not{p})} \tag{21}$$

where

$$\Sigma(\not{p}) = \Sigma_2(\not{p}) - (\delta_m + \delta_2)m + \mathcal{O}(e^4)\,. \tag{22}$$

Calculate the one-loop electron self energy, i.e. $\Sigma_2(\not{p})$, and show that[3]

$$\delta_2 = -\frac{\alpha}{4\pi}\frac{1}{\bar{\epsilon}} \tag{23}$$

$$\delta_m = -\frac{3\alpha}{4\pi}\frac{1}{\bar{\epsilon}}\,, \tag{24}$$

in the $\overline{MS}$ scheme. With these counterterms, you should find the UV-finite expression

$$\Sigma(\not{p}) = \frac{\alpha}{4\pi}\left(2m - \not{p} + 2\int_0^1 dx\Big[(1-x)\not{p} - 2m\Big]\ln\frac{\mu^2}{\Delta}\right) + \mathcal{O}(e^4) \tag{25}$$

where

$$\Delta = xm^2 - x(1-x)p^2\,. \tag{26}$$

Now, there is an IR-divergence in this expression, but we will not worry about these now. The easy way to cure this divergence is to add a photon mass to the photon propagator, but let us not venture into these details at this point.

[3] Remember that $\bar{\epsilon}$ includes $\ln 4\pi$ and γ_E.