Problem sheet 6

FYS5120-Advanced Quantum Field Theory

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⇝ These problems are scheduled for discussion on **Wednesday, 30 February 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

In this problem sheet, we will study redundant systems and how to treat them using the so-called gauge fixing procedure. We will see how this is done in a two-dimensional system with rotational invariance before turning to gauge-invariant theories. After removing the redundancy in our description, the resulting theory has a residual symmetry called BRST-symmetry. This symmetry has a sophisticated mathematical foundation and is central when studying, e.g. string theory, supergravity and quantization of gravity. Last but not least, it is an instrumental tool in showing the renormalizability of non-Abelian gauge theories. Thus, it is important to familiarize yourself with the basic concepts of BRST.

Problem 12: Finite-Dimensional Gauge Fixing and BRST-invariance

To get physical results as, for instance, to evaluate S-matrix elements, it is necessary to integrate over gauge-invariant functionals

$$\langle F[A]\rangle = \int_{\mathcal{A}} \mathcal{D}A\, F[A]\, e^{iS[A]}\,, \tag{1}$$

where $\mathcal{A}$ the space of all connections (gauge fields). But, if we insist on gauge invariance, then it follows that the integrand is invariant along gauge orbits. The gauge group orbits are defined as the sets of points in the field space, which can be reached from a given A via a gauge transformation. In other words, given A, its orbit is given by all fields A^g obtained by varying the parameters of the gauge group. Since the set of orbits defines equivalence classes, we can imagine the field space as the set of all the orbits. Correspondingly, we can divide our functional integral in a part parallel and in a part perpendicular to the orbits. Then, our functional integral is not well defined because the integrand along the direction parallel to the orbits is invariant, implying that the integral is infinite. But, this observation suggests a simple solution to the problem. We could define the integral by dividing it by the integral along the orbits, which is the volume of the group of all gauge transformations. Thus, the objective is to find a generating functional on the form[1]

$$\mathcal{Z} = \int_{\mathcal{A}/\mathcal{G}} d\mu(A) e^{-S[A]}\,, \tag{2}$$

where $\mathcal{G}$ is the space of all gauge transformations, so that $\mathcal{A}/\mathcal{G}$ denotes the space of all gauge equivalence classes of connections.

In order to gain a clear understanding of this procedure, let us consider the following simpler example, in two dimensions

$$\mathcal{Z} = \int_{\mathbb{R}^2} dx\, e^{-S(x,y)}\,, \tag{3}$$

[1] In this expression, we consider the Euclidean version of the path integral.

a) Let us think of $S(x,y)$ as playing the role of our 'action' for 'fields' (x,y), while rotations represents 'gauge transformations' leaving this integral invariant. Show that rotational invariance implies

$$\mathcal{Z} = \mathrm{vol}(SO(2)) \int_0^\infty dr\, r\, e^{-S(r)} \,. \tag{4}$$

The idea being here that for each fixed r, the finite[2] volume $\mathrm{vol}(SO(2))$ denotes the redundancy in $\mathcal{Z}$. In general the group of gauge transformations has infinite volume, so we would like to express the generating functional in such a way that that the volume of the group is absent.

b) Draw the space of gauge orbits and indicate how you would implement a gauge fixing surface that only intersects the orbits once. What is the space $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$ in this context? In general, the space of connections $\mathcal{A}$ is an affine space. If we wanted to model an affine space in this context, what would the form of $\mathcal{A}/\mathcal{G}$ be?

c) Use the Faddeev-Popov prescription to gauge fix this integral, i.e. specify a gauge fixing function $f(r,\theta)$, and show that you can write

$$\mathcal{Z} = \int d\mu\, e^{-S_{eff}} \,, \tag{5}$$

where

$$S_{eff} = S(r) + \frac{1}{2\alpha}\big(f(r,\theta)\big)^2 + c^* M c \,, \tag{6}$$

for some specific M, and (c^*, c) a pair of grassmann variables.

d) Integrate in a Lagrange multiplier h and show that

$$S_{eff} = S(r) + \frac{\alpha}{2}h^2 - ihf(r,\theta) + c^* M c \,. \tag{7}$$

Write down the concrete expressions of the BRST transformations $\delta_B \Phi = \zeta(Q\Phi)$, where the Grassmann variable ζ parametrizes the transformation and $\Phi = (x_i, c^*, c)$. Show that S_{eff} is invariant under these transformations and that $Q^2 = 0$, i.e. Q is *nilpotent*.

Hint: You may find it useful that the BRST transformation Q has to obey the generalized Leibnitz rule

$$Q(FG) = (QF)G + (-1)^{f_F} F(QG) \,, \tag{8}$$

where f_F is the Grassmann parity of F. An object which is Grassmann odd has Grassmann parity 1, and 0 otherwise. In other words, if F is bosonic $f_F = 0$ and if F is fermionic $f_F = 1$.

e) Based on the previous exercise, show that we can write the generating functional as

$$\mathcal{Z} = \int d\mu\, e^{-S(r) + Q\Psi} \,, \tag{9}$$

where the BRST invariance of the full action is apparent.

[2] The volume of any compact Lie group is finite. See [1] for a systematic calculation of volumes of compact manifolds.

f) For those of you that want more practice: do similar steps as above for the integral we studied in the group sessions, i.e.

$$\mathcal{Z} = \int_{\mathbb{R}^2} dx\, e^{-S(x,y)}\,, \tag{10}$$

where the 'action' is

$$S(x,y) = \frac{1}{2}(x-y)^2\,, \tag{11}$$

and translations play the role of gauge transformations.

Problem 13: BRST-invariance for Gauge Theories

By performing an analogous analysis as above, convince yourself that the generating functional for a non-Abelian gauge theory can be written as

$$\mathcal{Z} = \int \mathcal{D}\mu\, e^{iS_{eff}[A]}\,, \tag{12}$$

where (including fermions)

$$S_{eff}[A] = S_{YM}[A] + S_D[\psi, A] + \frac{\xi}{2}(h^a)^2 + h^a F^a[A] + \bar{c}^a M^{ab} c^b\,, \tag{13}$$

where

$$F^a[A] = \partial^\mu A^a_\mu \tag{14}$$

$$M^{ab} = \frac{\delta F^a}{\delta \alpha^b}\,. \tag{15}$$

a) Show that the effective Yang-Mills Lagrangian is invariant under the BRST transformations

$$\delta_B A^a_\mu := \zeta(Q A^a_\mu) = \zeta D^{ab}_\mu c^b \tag{16}$$

$$\delta_B \psi := \zeta(Q\psi) = ig\zeta c^a t^a \psi \tag{17}$$

$$\delta_B c^a := \zeta(Q c^a) = -\frac{1}{2} g\zeta f^{abc} c^b c^c \tag{18}$$

$$\delta_B \bar{c}^a := \zeta(Q\bar{c}^a) = \zeta h^a \tag{19}$$

$$\delta_B h^a := \zeta(Q h^a) = 0\,, \tag{20}$$

where again ζ is a Grassmann variable parametrizing the transformation.

b) Show, by means of explicit calculations, that $Q^2\Phi$ vanishes for all fields Φ, and that

$$S_{eff}[A] = S_{YM}[A] + S_D[\psi, A] + Q\Big(\bar{c}^a \partial^\mu A^a_\mu + \frac{\xi}{2}\bar{c}^a h^a\Big)\,, \tag{21}$$

manifesting the BRST invariance of the full gauge fixed action.

c) Discuss the implications of the nilpotency of Q.

References

[1] L. J. Boya, E. Sudarshan and T. Tilma, *Volumes of compact manifolds*, *Reports on Mathematical Physics* **52** (Dec, 2003) 401–422.