Problem sheet 7

FYS5120-Advanced Quantum Field Theory

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$\rightsquigarrow$ These problems are scheduled for discussion on **Wednesday, 30 February 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

We will study a simple use of so-called differential forms in this problem set. We will explore how the language of differential forms powerfully reduces Maxwell's equations to two simple equations. More importantly, we will see how to construct Lagrangians out of forms, given the arena of four-dimensional manifolds. The Lagrangians we are left with are known to you, but the idea here is to introduce you to the notion of differential geometry in QFT. Also, we will meet one type of Lagrangian here that will work as a first peek into topological quantum field theory.

Problem 14: Differential Forms and Invariant Lagrangians

In this exercise, we will study the use of differential forms. Before that, it is important to stress that this exercise is not strictly necessary for QFT, meaning that I will not test you in differential forms or any other notions from differential geometry. However, it is important to know that gauge field theory has a natural geometric formulation.

a) Let us consider a flat space with Euclidean metric, e.g. $(\mathbb{R}^2, \delta)$ (or $(\mathbb{R}^3, \delta)$). Consider a vector potential $\mathbf{A}$, a 0-form φ, a 1-form potential A, and use the Hodge star $\star$ and exterior derivative d to show that

$$\star dA = \nabla \times \mathbf{A} \tag{1}$$

$$\star d \star A = \nabla \cdot \mathbf{A} \tag{2}$$

$$\star d \star d\varphi = \nabla^2 \varphi \,. \tag{3}$$

b) Let us next consider flat space with non-Euclidean metric, e.g. $(\mathbb{R}^2, g)$ using polar coordinates and show that the formulas above remain the same.

Remark: A space with Euclidean or pseudo-Euclidean metric is called flat, and the coordinates in which the metric is (pseudo) Euclidean is also often called flat. For example, the Minkowski space of special relativity is flat. The coordinates are called curvilinear if the metric is not everywhere (pseudo) Euclidean. For example, in the flat 2-dimensional space with polar coordinates r, θ the metric is non-Euclidean,

$$ds^2 = dr^2 + r^2 d\theta^2 \,, \tag{4}$$

However, if a coordinate transformation exists that globally (that is, everywhere) turns the metric into (pseudo) Euclidean, the space is still called flat. In our example, such a *transformation is*

$$x \to r\cos\theta \tag{5}$$
$$y \to r\sin\theta \tag{6}$$

If such transformation does not exist, the space is called curved or Riemann space. The geometry of a curved space is non-Euclidean or Riemann geometry, which some of you will meet in GR this semester.

c) Show that the exterior derivative is nilpotent, i.e. $d^2 = 0$. Further, use the nilpotency of d and that the field strength F is a 2-form to show that

$$dF = 0 \tag{7}$$
$$d\star F = 0\,, \tag{8}$$

which are the differential form of the homogenous Maxwell equations.

d) Define the inner product of two r-forms and use it to show that the (Abelian) Maxwell action can be written as

$$S_1[A] = a_1 \int_M F\wedge\star F = -\frac{1}{4}\int_M d^4x\, F_{\mu\nu}F^{\mu\nu} \tag{9}$$

where M is the four-dimensional spacetime manifold with Minkowski metric, which we often write as (M, g).

e) In principle, the following term can be added to the Maxwell action

$$S_2[A] = a_2 \int_M F\wedge F\,. \tag{10}$$

Discuss how the term transform under C, P, T transformations. Argue why we at first glance can drop such terms.

Hint: The simplest answer for $S_2[A]$ comes from the generalized Stoke's theorem. Another way is to write it out on component form where the answer should become apparent after a short rewriting. We will actually meet this action later in the course, but then in the disguise of what we call Chern-Simons theories.

f) Consider next the following terms

$$S_3[A] = a_3 \int_M A\wedge A\wedge A\wedge A + a_4\int_M A\wedge A\wedge F\,, \tag{11}$$

and argue why these terms can be dropped from our analysis despite having the correct form structure in four dimensions.

g) Let do the same analysis for Yang-Mills theory: define the inner product of Lie algebra valued forms to construct an invariant action in four dimensions and write the equations of motions in form structure[1]. Further, discuss the Yang-Mills version of $S_2[A]$.

[1]If we have time later in the course, we will analyze a couple of interesting solutions to these equations.