# Problem sheet 8

## FYS5120-Advanced Quantum Field Theory

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⇝ These problems are scheduled for discussion on **Wednesday, 23 February 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

In this problem sheet, we will finally explore higher-order corrections and introduce the notion of regularization and renormalization. As discussed in the group sessions, the naive approach of calculating loop diagrams leads to nonsense, i.e., we encounter divergent integrals. A simple way of tackling such divergences is to cut off the momentum integral sharply. However, eventually we want regularize in such a way that all the symmetries of the theory is preserved. Thus, we will consider Pauli-Villars regularization which preserves all symmetries in Abelian theories.

First, we will first study the short distance (high energy) behaviour of the scalar two-point function $\langle\phi(x)\phi(y)\rangle_0$. This analysis will lead us to conclude that our naive approach in calculating loop diagrams is mathematically ill-defined, implying that we need to revisit the definition of propagators. Further, we will spend some tume building up for a full covering of Wilsonian renormalization. However, before such an analysis we have to visit some technical details and techniques.

### Problem 14: Singularities on the Lightcone

As a warm up for vacuum polarization in gauge theory, let us consider the simple $\phi^3$-theory[1] with (Minkowski signature) action

$$S[\phi] = S_G[\phi] + S_{\text{int}}[\phi] \tag{1}$$

$$S_G[\phi] = -\frac{1}{2}\int d^4x\, d^4y\, \phi(x)K(x,y)\phi(y)\,. \tag{2}$$

$$S_{\text{int}}[\phi] = \frac{\lambda}{3!}\int d^4x\, \phi^3(x) \tag{3}$$

where

$$K(x,y) = \delta^{(4)}(x-y)(\Box_y + m^2)\,. \tag{4}$$

Let us define the scalar two-point function as

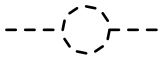
$$\langle\phi(x)\phi(y)\rangle := \Delta(x-y)\,. \tag{5}$$

To one-loop order we have the following momentum space expansion

$$\Delta(k) = \Delta_0(k) + \Delta_0(k)\big[i\Pi(k)\big]\Delta_0(k) + \cdots \tag{6}$$

[1] There are numerous problems with such a theory, but instead of worrying about details we only want to study the issue of infinities.

where

$$i\Pi(k) =$$  $$= \frac{\lambda^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p-k)^2 - m^2 + i\epsilon}\frac{1}{p^2 - m^2 + i\epsilon} \tag{7}$$

a) Show that the one loop correction has a logarithmic divergence. Introduce Feynman parameters and use the identity

$$\frac{1}{AB} = \int_0^1 dx\, dy \delta(x+y-1)\frac{1}{(xA+yB)^2} \tag{8}$$

to rewrite the loop correction on a form more suitable for evaluation.

b) Instead of trying to evaluate this integral right away, let us investigate the behaviour of $\Delta_0(x)$ at short distances and show that it is not surprising the loop integrals as above are divergent.

First, show that the coordinate representation of two-point function can be written as

$$\Delta_0(x) = \theta(x^2)\frac{im}{8\pi\sqrt{x^2}}H_1^{(1)}(m\sqrt{x^2}) + \theta(-x^2)\frac{m}{4\pi^2\sqrt{-x^2}}K_1(m\sqrt{-x^2}) - \frac{i}{4\pi}\delta(x^2) \tag{9}$$

where $H_\nu^{(1)}$ is a Hankel function and $K_1$ is a modified Bessel function[2].

c) Show that in the limit $m \to 0$, we get

$$\Delta_0(x) = \frac{1}{4\pi^2}\frac{1}{x^2 + i\epsilon} \tag{10}$$

which clearly diverges on the lightcone.

*Hint: To obtain the expression in Eq. (10), it will be helpful to use the identity*

$$\lim_{\epsilon\to 0}\frac{1}{s+i\epsilon} = PV\,\frac{1}{s} - i\pi\delta(s) \tag{11}$$

*for distributions.*

d) We have established that causal Green's functions are singular functions with singularities on the lightcone. From a mathematical point of view they represent distributions. Thus, since loop integrals like Eq. (7) contains squares of singular distributions of the form $\delta(x^2)$ and $x^{-2}$, they do not represent well-defined quantities. In fact, the necessity of a special definition of products is, on the whole, typical for distributions. Hence, the divergence in loop integrals is related to the fact that expectation values of product of fields at coinciding points (local monomials) are always divergent.

To make this well defined we have to regularize the distribution[3]

$$\Delta_0(x) \to \Delta_0^{\text{reg}}(x)\,. \tag{12}$$

[2] Bessel functions satisfy a wealth of identities. For more information, see e.g. Arfken and Weber's Mathematical Methods for Physicists.

[3] For those interested in learning more about regularization of distributions in QFT, see [1].

Let us consider Pauli-Villars regularization[4], defined as

$$\Delta_0^{\text{reg}}(k) = \frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - \Lambda_0^2 + i\epsilon} \tag{13}$$

where $\Lambda_0$ is the ultraviolet cutoff. Use the regularized version of the propagator to show that the loop integral take the form

$$\Pi(k) = -\frac{\lambda^2}{32\pi^2} \int_0^1 dx \, \ln\left( \frac{m^2 - k^2 x(1-x)}{\Lambda_0^2} \right) \tag{14}$$

which is finite as long as $\Lambda_0$ is kept finite. However, we want to get rid of the arbitrary scale $\Lambda_0$, and this procedure is what we call renormalization. However, we are not going to focus on magically cancelling infinities. The modern view of renormalization is due to Kenneth Wilson and this is the method we will focus on.

## Problem 15: A Toy Model

Before treating the full picture of Wilsonian renormalization, let us take a few steps back in order to point out some technical issues regarding perturbation theory[5].

Let us consider the simplest possible set-up we can think of, namely QFT in zero dimensions. This might seem like an absurd thing to do. However, it turns out that the zero-dimensional case is crucial to understand because we can do things without worrying about infinite-dimensional spaces.

If our 'spacetime' is zero-dimensional and connected, then it follows that it must be just a single point. Hence, our 'fields' in this universe can be thought of as maps

$$\phi : \{pt\} \to \mathbb{R} \,, \tag{15}$$

i.e. just a real variable. Notice that the Lorentz group is trivial in zero dimensions, and the notion of spin is absent. More obviously, there are no spacetime directions to differentiate our fields, so there are no kinetic terms. Similarly, the configuration space is $\mathcal{C} \cong \mathbb{R}$, which is finite-dimensional. Thus, the action is just a normal function

$$S : \mathcal{C} \cong \mathbb{R} \to \mathbb{R} \tag{16}$$

of one real variable. Consequently, the path integral measure $\mathcal{D}\phi$ can be taken to be the standard Lebesgue measure $d\phi$ on $\mathbb{R}$.

Take the action to be

$$S(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \,, \qquad \lambda > 0 \,, \;\; m^2 > 0 \tag{17}$$

giving the generating function[6]

$$\mathcal{Z} = \int_{\mathbb{R}} d\phi \, e^{-S(\phi)/\hbar} \tag{18}$$

The method of treating these integrals is to use a formal power series expansion, which we represent graphically using Feynman diagrams. However, the best we can do is to obtain an *asymptotic expansion* for such integrals. This is because the integral cannot have a Taylor

[4] Instead of imposing a sharp cutoff; we have regularized more dynamically. A sharp momentum cutoff will not preserve the symmetries of the theory, and thus we have to consider more sophisticated methods. It turns out that Pauli-Villars work for Abelian theories, but for non-Abelian we need another method.

[5] Recommend reading Chapter 1 and Chapter 5 of [2] before doing this exercise.

[6] Here we restore $\hbar$ where it would appear in the path integral (simple dimensional analysis). We will later show that the expansion in $\hbar$ is equivalent to the expansion in the coupling that you are more familiar with.

expansion around $\lambda = 0$ (or equivalently $\hbar = 0$), since any such Taylor expansion would have to converge for all $\lambda$ (or $\hbar$) around a disc $D \subset \mathbb{C}$ centered on the origin. Hence, perturbation theory tells us important, but not complete, information about our QFT.

a) Let us show that the series is asymptotic by an explicit calculation: expand the generating functional and show that

$$\mathcal{Z} \sim \mathcal{Z}_0 \left[ 1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} + \cdots \right] \tag{19}$$

where $\sim$ denotes that this is an asymptotic expansion and $Z_0 = \sqrt{2\pi\hbar}/m$.

*Hint: Be careful when interchanging summation and integration.*

b) Now, use the Feynman diagram technique to reproduce the same expression. You should find that

$$\mathcal{Z}/\mathcal{Z}_0 \sim \sum_i G_i \tag{20}$$

$$= 1 - \frac{1}{8}\frac{\hbar\lambda}{m^4} + \frac{1}{48}\frac{\hbar^2\lambda^2}{m^8} + \frac{1}{16}\frac{\hbar^2\lambda^2}{m^8} + \frac{1}{128}\frac{\hbar^2\lambda^2}{m^8} + \cdots \tag{21}$$

where the numerical factor in front is known as the symmetry factor of each graph, i.e. the graph automorphism Aut(G)[7].

c) The *linked cluster theorem* tells us that the connected graphs[8] (those that end up contributing in scattering processes) is given by

$$\mathcal{W} := \ln \mathcal{Z} \tag{22}$$

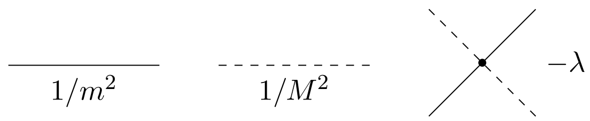
while in QFT, this is also known as the *Wilsonian effective action*, and is closely related to the Helmholtz free energy in statistical physics. Based on this, write the diagrammatic form of $\mathcal{W}$ up to $\mathcal{O}(\lambda^2)$ for the theory considered.

## Problem 16: An Effective Theory

Suppose we have two real-valued 'fields' $\phi$ and $\chi$ so that the space of fields is $\mathbb{R}^2$. We consider the action

$$S(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}M^2\chi^2 + \frac{\lambda}{4}\phi^2\chi^2\,, \tag{23}$$

From this action we have the Feynman rules[9],




[7] In the mathematical field of graph theory, an automorphism of a graph is a form of symmetry in which the graph is mapped onto itself while preserving the edge–vertex connectivity.

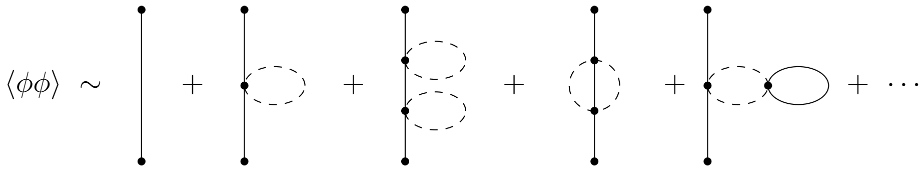
[8] Peskin-Schroeder spend an awful lot of time explaining why we can cancel bubble and disconnected graphs, but the linked cluster theorem is all we need.

[9] From now on we can just set $\hbar = 1$.

which may be used to compute perturbative expression for correlation functions such as

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} d\phi d\chi \, e^{-S(\phi,\chi)} \, f(\phi, \chi) \, . \tag{24}$$

For example, we can calculate $\langle \phi\phi \rangle$ as follows




$$\frac{1}{m^2} - \frac{\lambda}{2m^4 M^2} + \frac{\lambda^2}{4m^6 M^4} + \frac{\lambda^2}{2m^6 M^4} + \frac{\lambda^2}{4m^6 M^4} + \cdots$$

Instead of using Feynman diagrams, let us try to arrive at this result in a different way. Suppose we have no idea what the properties of $\chi$ are, e.g. $\chi$ is so heavy that it is intractable in our experiments, and we can only measure properties of $\phi$. Since we have no idea of what $\chi$ is doing, this suggests that we can integrate it out and keep $\phi$ fixed. From this point of view, whilst performing the $\chi$ integral, the term $\phi^2\chi^2$ acts as a source ($J = \phi^2$) term for the 'operator' $\chi^2$.

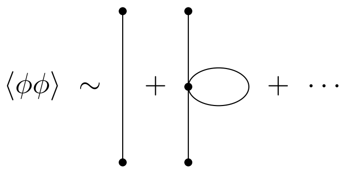
a) Integrate out the high energy field $\chi$ and show that the effective action takes the form

$$S_{\text{eff}} = -\ln \Big[ \int_{\mathbb{R}} d\chi \, e^{-S(\phi,\chi)} \Big] \tag{25}$$

$$:= \frac{m_{\text{eff}}^2(M)}{2}\phi^2 + \frac{\lambda_4(M)}{4!}\phi^4 + \frac{\lambda_6(M)}{6!}\phi^6 + \cdots \, , \tag{26}$$

where the new couplings become dependent on the large mass $M$.

b) Define the effective Feynman rules and show that we only need two diagrams




to reproduce $\langle \phi\phi \rangle$ in the effective theory (up to $\mathcal{O}(\lambda^2)$)

This might not seem very impressive as it is in general very difficult to compute $S_{\text{eff}}$. However, once we have the effective action, we arrived at this answer using just two diagrams, whereas above, it required five. Thus, in certain cases, it can save much computational work. However, the real point of this whole discussion is this: the way we actually perceive the world is through $S_{\text{eff}}$. In general, we have no idea what new physics may be lurking beyond the current reach of our experiments. Thus, in describing low-energy physics, the main objective should be to describe the behaviour of the degrees of freedom that are relevant and accessible at the energy scale at which the experiments are conducted, even when we know what the more fundamental description is.

# References

[1] A. Aste, C. von Arx and G. Scharf, *Regularization in quantum field theory from the causal point of view*, *Progress in Particle and Nuclear Physics* **64** (Jan, 2010) 61–119.

[2] R. E. Borcherds and A. Barnard, *Lectures on quantum field theory*, 2002.