Problem sheet 9

FYS5120-Advanced Quantum Field Theory

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$\rightsquigarrow$ These problems are scheduled for discussion on **Wednesday, 06 April 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

Problem 17: Wilsonian Renormalization

In this exercise, we will carry out the procedure of Wilson renormalization for a four-dimensional scalar theory. Wilson's treatment develops an important statement: field theories describe physics only in a limited range of energies. This is the starting point for the modern way of understanding renormalization, where we can consider a QFT as an effective field theory[1].

To implement the idea, let us consider a QFT with a physical UV cutoff Λ_0. For energies above Λ_0 physics is described by another theory (string theory, another QFT, . . .), which is assumed to be fundamental, while below Λ_0 it is represented by a general (bare) action which contains in principle all possible interaction terms compatible with the symmetries of the theory. Suppose our QFT is governed by the action

$$S_{\Lambda_0}[\phi] = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 + \sum_i g_i \mathcal{O}_i(x)\right] \tag{1}$$

where we have allowed for arbitrary local operators $\mathcal{O}_i(x)$ subject to two symmetries, the Euclidean group[2] $ISO(4)$ and the discrete symmetry $\mathbb{Z}_2$. Given this action, we have a regularized generating functional

$$\mathcal{Z}_{\Lambda_0}(g_i) = \int_{\mathcal{C}^\infty(M)_{\leq\Lambda_0}} \mathcal{D}\phi \, e^{-S_{\Lambda_0}[\phi]}\,, \tag{2}$$

where the integral is taken over the space $\mathcal{C}^\infty(M)_{\leq\Lambda_0}$ of smooth functions on M whose energy is at most Λ_0.

a) Now let us think what happens as we try to perform the path integral by first integrating those modes with energy between $(\Lambda, \Lambda_0]$: split ϕ into high-energy ϕ_H and low-energy ϕ_L modes and show that we can formally write

$$S^{\text{eff}}_{\Lambda}[\phi_L] = -\ln\left[\int_{\mathcal{C}^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\phi_H \, e^{-S_{\Lambda_0}[\phi_L+\phi_H]}\right] \tag{3}$$

Recalling from *Problem set 8* we see that this is exactly the Wilsonian effective action where the low energy fields serve as sources for the high energy fields, which are the only

[1] Here I should mention that the axiomatic QFT community disagrees with this statement, but this is far beyond what we will cover in this course.

[2] In Lorentzian signature this would obviously be the Poincare group.

ones that can propagate. Therefore, we can calculate this object perturbatively by summing over all connected Feynman diagrams with a set number of external fields ϕ_L, and the only trace of the integrated fields ϕ_H is that they appear in loops.

b) To make the procedure clear let us consider only the first few terms in S_{Λ_0}, i.e. we consider

$$S_{\Lambda_0}[\phi] = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \cdots\right] \tag{4}$$

Integrate out the high energy modes, i.e. perform the loop diagrams for ϕ_H, and show that we obtain the effective action

$$S^{\text{eff}}_{\Lambda}[\phi] = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\tilde{m}^2\phi^2 + \frac{\tilde{\lambda}}{4!}\phi^4 + \frac{B^\mu}{4!}\phi^3\partial_\mu\phi + \frac{C^{\mu\nu}}{4!}\phi^2\partial_\mu + \cdots\right] \tag{5}$$

Recall that the bare action we started with involved all possible terms satisfying the imposed symmetries. Thus, the effective action we have obtained has the same form as the one we started with but with different coefficients, i.e.

$$S^{\text{eff}}_{\Lambda}[\phi] = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 + \sum_i \tilde{g}_i(\Lambda)\mathcal{O}_i(x)\right]. \tag{6}$$

[*Hint: To find the correction to m and λ, the loop integrals to calculate are*

$$-\frac{\lambda}{4}\int d^4x\, \phi_L^2(x)\, \langle\phi_H(x)\phi_H(x)\rangle_0 \tag{7}$$

and

$$-\frac{\lambda^2}{16}\int d^4x d^4y\, \phi_L^2(x)\phi_L^2(y)\, \langle\phi_H(x)\phi_H(y)\rangle_0{}^2 \tag{8}$$

where the last one is non-local, i.e. ϕ_L is evaluated at different points, which cannot appear as corrections of a local bare action. To overcome this problem expand the non-local terms in a series of infinite local contributions with growing number of derivatives of the field. In such a way the corrections arising from them can be included in the action as an infinite number of derivative interactions.]

c) Argue that the generating functional

$$Z_\Lambda(\tilde{g}_i(\Lambda)) = \int_{C^\infty(M)_{\leq\Lambda}} \mathcal{D}\phi\, e^{-S^{\text{eff}}_{\Lambda}[\phi]} \tag{9}$$

obtained from the effective action scale Λ is exactly the same as we started with, i.e.

$$Z_\Lambda(\tilde{g}_i(\Lambda)) = Z_{\Lambda_0}(g_i, \Lambda_0) \tag{10}$$

Hence, show that we obtain the following differential equation

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \Lambda\frac{\partial g_i(\Lambda)}{\partial\Lambda}\frac{\partial}{\partial g_i}\right)\mathcal{Z}_\Lambda(\tilde{g}) = 0 \tag{11}$$

which is commonly called the *renormalization group equation.*

Comment: Integrating out high energy modes is cumbersome and can be quite tricky to do for more advanced theories. However, Joe Polchinski and Christof Wetterich have *developed methods building on Wilson's ideas. These methods are easier to work with and are known as the exact renormalization group (Polchinski) and functional renormalization group (Wetterich)*[3]*. Polchinski used his method to show that the Wilsonian renormalization approach was equivalent to the traditional approach laid out by Freeman Dyson et al. in QED (called continuum renormalization). Thus, while the continuum approach is easier to work with in practice, it is the Wilsonian idea we have in mind.*

d) Let us introduce dimensionless couplings[4] in the following way

$$\tilde{\lambda}_i = \Lambda^{-[\tilde{g}_i]}\tilde{g}_i \tag{12}$$

where $[\tilde{g}_i]$ denotes the mass dimension of the couplings and Λ is the current scale of the theory. Use the renormalization group equation to calculate the *beta functions*

$$\beta_2 := \Lambda \frac{\partial \tilde{\lambda}_2}{\partial \Lambda} \tag{13}$$

$$\beta_4 := \Lambda \frac{\partial \tilde{\lambda}_4}{\partial \Lambda} \tag{14}$$

and decide $\operatorname{sgn}(\beta_i)$, i.e. the sign of the beta functions. Discuss the connection between $\operatorname{sgn}(\beta_i)$ and the notion of renormalization group flow, thus deciding the relevance of the possible terms in the Lagrangian.

[*Hint: The conclusion of your analysis should be that the only relevant terms are*

$$S[\phi] = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right] \tag{15}$$

i.e. all other terms are irrelevant. In the terminology of renormalization theory, it is called renormalizable, and in the RG terminology, marginal.]

Problem 18: The Continuum Limit and Perturbative Renormalization

So far, we have considered a fixed initial theory $S_{\Lambda_0}[\phi]$ with initial couplings g_i and examined how these couplings change as we probe the theory at long distances. Our definition of the effective action ensured that the generating function and hence the correlation functions are independent of the low-scale Λ. What about the cutoff scale Λ_0? Suppose we fix a particular low–energy theory (perhaps motivated by the results of some finite–scale experiments). How can we remove the high–energy cut–off, sending $\Lambda_0 \to \infty$, without affecting what the theory predicts for low–energy phenomena. We call this taking the continuum limit of our theory since sending $\Lambda_0 \to \infty$ is allowing the field to fluctuate on arbitrarily small scales.

The key to achieving this is due to the universality of the renormalization group flow. It assures us that the properties of the theory in the IR are determined not by the infinite set of couplings $\{g_i\}$, but only by the couplings to a few relevant operators. Hence, it is possible to tune the initial couplings so that the theory remains finite as $\Lambda_0 \to \infty$. We can achieve this tuning by rewriting the bare action as

$$S_{\Lambda_0}[\phi] \to S_{\Lambda_0}[\phi] + S^{\mathrm{CT}}_{\Lambda_0}[\phi] \tag{16}$$

[3] The latter is just a Legendre transform of the former

[4] This might seem as a strange thing to do, but the real motivation will reveal itself in a couple of weeks. That is, there is a certain way of classifying renormalizable theories and the dimension of the coupling is important in that regard.

where the counterterms (CT) are tuned so that the limit

$$e^{-S^{\text{eff}}_{\Lambda}[\phi_L]} = \lim_{\Lambda_0\to\infty}\left[\int_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\phi_H\, e^{-S_{\Lambda_0}[\phi_L+\phi_H]-S^{\text{CT}}_{\Lambda_0}[\phi_L+\phi_H]}\right] \tag{17}$$

exists, provided we take this limit after computing the path integral. Notice that nothing new has been added to the theory, i.e. the original action we started with contained all possible monomials in fields and their derivatives. The reason for making S^{CT} explicit, rather than just treating the counterterms as a modification of $\{g_i\}$, is that in practice we work perturbatively. To evaluate the path integral in Eq. (17), we first compute quantum corrections to, e.g. 1-loop order, using the original action S_{Λ_0}. These 1-loop corrections will depend on the cut–off Λ_0. In general, they will diverge as $\Lambda_0\to\infty$ reflecting the fact that we lose control of the original theory if the cut–off is removed naively. However, vertices in $S^{\text{CT}}_{\Lambda_0}$ provide further contributions to these quantum corrections. By tuning the values of the couplings in $S^{\text{CT}}_{\Lambda_0}$ by hand, we can obtain a finite limit.

Let us investigate one way of how this can be done in practice: Split the action in the following way

$$S_{\Lambda_0}[\phi] = \int d^4x\left[\frac{1}{2}(\partial\phi)^2+\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4\right] \tag{18}$$

$$S^{\text{CT}}_{\Lambda_0} = \int d^4x\left[\frac{1}{2}\delta Z_\phi(\partial\phi)^2+\frac{1}{2}\delta m^2\phi^2+\frac{\delta\lambda}{4!}\phi^4\right] \tag{19}$$

with $(\delta Z_\phi, \delta m^2, \delta\lambda)$ representing our freedom to adjust the couplings in the original theory (including the coupling to the kinetic term).

a) Define what is meant by a 1PI (one-particle irreducible) graph, and show that the full 2-point function takes the form

$$\Delta(p^2) = \frac{1}{p^2+m^2-\Pi(p^2)} \tag{20}$$

where $\Pi(p^2)$ are the 1PI.

b) Calculate the two-point function to one-loop order and show that[5]

$$\Pi^{\text{1-loop}}(p^2) = -\frac{\lambda}{32\pi^2}\left(\Lambda_0 - m^2\ln\left(1+\frac{\Lambda_0^2}{m^2}\right)\right) - p^2\delta Z_\phi - \delta m^2 \tag{21}$$

c) It is finally time to go to experiments to fix δZ_ϕ and δm^2. Experimentally, we can measure the true mass of a particle by looking for peaks (resonances) in scattering cross-sections where this particle is exchanged. These peaks correspond to poles of the S-matrix in the complex momentum plane. Let us then motivate the requirement of fixing δZ_ϕ and δm^2 by the fact that the classical (non-loop) propagator $(p^2+m^2)^{-1}$ has a pole when $-p^2 = m^2$. Hence, a reasonable condition for the *exact* ϕ propagator $\Delta(p^2)$, is that is has a simple pole at some experimentally measured value $-p^2 = m^2_{\text{phys}}$, and that the residue of this pole is unity. Thus, we demand that

$$\Pi(-m^2_{\text{phys}}) = m^2 - m^2_{\text{phys}} \tag{22}$$

$$\left.\frac{\partial\Pi}{\partial p^2}\right|_{p^2=-m^2_{\text{phys}}} = 0 \tag{23}$$

[5]Note that we use Euclidean signature.

Show that this procedure gives a perfectly well-behaved $\Pi(p^2)$ (up to 1-loop) in the continuum limit. Further, calculate the beta-function β_2 and relate it to the one you found in the previous exercise.

d) For those who want more practice, do a similar analysis of the correction to the coupling and show that we obtain a well-defined vertex function Γ (up to 1-loop) in the continuum limit. Further, find the beta function β_4 and discuss the scale behaviour of the coupling in the massless limit.