Problem sheet 4

FYS5120-Advanced Quantum Field Theory

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⇝ These problems are scheduled for discussion on **Wednesday, 16 February 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

Problem 9: Equations of motion

In the path integral formalism we only work with classical fields, so why do we get a "quantum" result? In this problem we will explore this using a general scalar field theory:

$$\mathcal{L}[\varphi] = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\text{int}}[\varphi]$$

a) Show that the equations of motion for this theory becomes

$$(\Box_x^2 + m^2)\varphi(x) = \frac{\partial \mathcal{L}_{\text{int}}}{\partial \varphi}[\varphi(x)]$$

by varying the action.

Now consider the generating functional

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi \exp\left(i\int d^4y \left(-\frac{1}{2}\varphi(y)\mathcal{O}_{\text{KG}}\varphi(y) + \mathcal{L}_{\text{int}}[\varphi(y)] + J(y)\varphi(y)\right)\right)$$

where $\mathcal{O}_{\text{KG}} = \Box_x^2 + m^2$ is the Klein Gordon differential operator. Since we are integrating over all field configurations $\mathcal{Z}[J]$ is invariant under a change

$$\varphi \to \varphi + \epsilon$$

for any field ϵ. By making this variation we get

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi \exp\left(i\int d^4y \left(-\frac{1}{2}(\varphi(y)+\epsilon(y))\mathcal{O}_{\text{KG}}(\varphi(y)+\epsilon(y)) + \mathcal{L}_{\text{int}}[\varphi(y)+\epsilon(y)] + J(y)\varphi(y) + J(y)\epsilon(y)\right)\right).$$

By expanding the relevant terms to first order in ϵ we get:

$$\mathcal{Z}[J] = \int \mathcal{D}\varphi \exp\left(i\int d^4y + \mathcal{L}[\varphi(y)] + J(y)\varphi(y)\right)$$
$$\left(1 + i\int d^4x \epsilon(x)\left(-\mathcal{O}_{\text{KG}}\varphi(x) + \frac{\partial \mathcal{L}_{\text{int}}}{\partial \varphi}[\varphi(x)] + J(x)\right) + \mathcal{O}(\epsilon^2)\right).$$

b) Fill in the gaps in the above calculation and show that

$$\mathcal{O}_{\text{KG}}\left(-i\frac{\delta \mathcal{Z}[J]}{\delta J(x)}\right) = \left(\frac{\partial \mathcal{L}_{\text{int}}}{\partial \varphi}\left[-i\frac{\delta}{\delta J(x)}\right] + J(x)\right)\mathcal{Z}[J].$$

This is known as the *Schwinger-Dyson differential equation*, and it uniquely determines the generating functional.

Consider now the expression

$$\left(-i\frac{\delta}{\delta J(x)}\right)\mathcal{Z}[J] = \int \mathcal{D}\varphi\varphi(x)\exp\left(i\int d^4y(\mathcal{L}[\varphi(y)] + J(y)\varphi(y))\right).$$

By making the variation $\varphi \to \varphi + \epsilon$ and expanding to first order in ϵ show that:

c)
$$(\mathcal{O}_{\text{KG}})_x\langle\varphi(x)\varphi(y)\rangle = -i\delta(x-y)$$
when $\mathcal{L}_{\text{int}} = 0$.

d)
$$(\mathcal{O}_{\text{KG}})_x\langle\varphi(x)\varphi(y)\rangle = \left\langle\frac{\partial\mathcal{L}_{\text{int}}}{\partial\varphi}[\varphi(x)]\varphi(y)\right\rangle - i\delta(x-y)$$
for a general $\mathcal{L}_{\text{int}} \neq 0$.

Note: Here $\langle ... \rangle$ is a shortening of $\langle\Omega| T\{...\} |\Omega\rangle$, also the notation $(\mathcal{O}_{KG})_x$ means that this is a differentiation with respect to the variable x.

These are all variations of the Schwinger-Dyson differential equation, and one can show using these equations that the canonical way of quantizing a field is equivalent to the path integral formalism! These are also what we think of as the equations of motion for the theory.

Problem 10: Non-Abelian Yang-Mills theories

Consider the general Yang-Mills Lagrangian for a fermion field charged under some gauge symmetry $\mathfrak{g}$:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}_i(i\not{D}_{ij} - m\delta_{ij})\psi_j$$

Let $\mathfrak{g} = \mathfrak{su}_3$, and i, j range over r, g, b.

a) Write out $\not{D}$ explicitly as a 3×3 matrix.
Recall: The generators of $\mathfrak{su}_3$ is given by $T^a = \frac{1}{2}\lambda^a$, where λ^a are the Gell-Mann matrices.

b) Write out the interaction terms contained in $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$.

c) When deriving the gauge boson propagator of a non-Abelian gauge theory we introduce Faddeev-Popov ghosts and anti-ghosts. If we include these ghosts fields in the Lagrangian the full Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu\bar{c}^a)(\delta^{ac}\partial^\mu + gf^{abc}A^{b\mu})c^c + \bar{\psi}_i(i\not{D}_{ij} - m\delta_{ij})\psi_j$$

With this derive the four vertex rules of this theory and draw the accompanying Feynman diagrams.