# Problem sheet 5

## FYS5120-Advanced Quantum Field Theory

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⇝ These problems are scheduled for discussion on **Wednesday, 23 February 2022**. If you spot any typos and/or mistakes please send an email to ***lasselb@fys.uio.no*** or ***jonaeid@math.uio.no***.

### Problem 11: Ward-Takahashi identities

Let

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

and consider the transformations

$$\psi(x) \to e^{-i\alpha(x)}\psi(x)$$
$$\bar{\psi}(x) \to e^{i\alpha(x)}\bar{\psi}(x).$$

As with the Schwinger-Dyson equations, the measures $\mathcal{D}\psi$ and $\mathcal{D}\bar{\psi}$ are invariant under such transformations.

a) By making the above transformations to the correlation function

$$\langle \psi(x)\bar{\psi}(y)\rangle = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\psi(x)\bar{\psi}(y)\exp\left(i\int d^4w\mathcal{L}\right)$$

and expanding to first order in $\alpha$ show that

$$\partial_\mu\langle j^\mu(z)\psi(x)\bar{\psi}(y)\rangle = -\delta(z-x)\langle\psi(x)\bar{\psi}(y)\rangle + \delta(z-y)\langle\psi(x)\bar{\psi}(y)\rangle$$

where $j^\mu(z) = \bar{\psi}(z)\gamma^\mu\psi(z)$ is the Noether current associated with charge conservation.

b) By applying the Fourier transform

$$\int d^4xd^4yd^4ze^{ipz}e^{ikx}e^{-iqy}$$

to the above equation show that

$$ip_\mu\mathcal{M}^\mu(p,k,q) = \mathcal{M}_0(k+p,q) - \mathcal{M}_0(k,q-p)$$

where $\mathcal{M}$ is the Fourier transform of $\langle...\rangle$.

This is known as the general Ward-Takahashi identity and it holds on the level of correlation functions. By using the LSZ-reduction formula one can prove the familiar Ward-identity for matrix elements:

If the matrix element $\mathcal{M} = \epsilon_\mu\mathcal{M}^\mu$ where $\epsilon_\mu$ is the polarization vector of an external photon leg, then

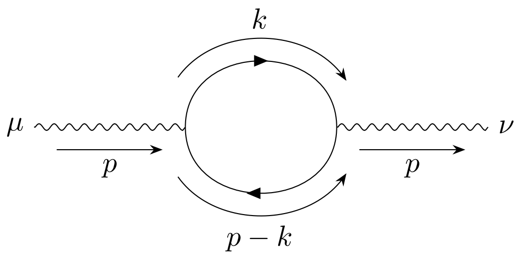
$$p_\mu\mathcal{M}^\mu = 0$$

where $p_\mu$ is the momentum of that photon.

c) Now consider the QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi.$$

Write down the matrix element $\mathcal{M}^{\mu\nu}$ of the second order corrections to the photon propagator:




i.e. don't contract with external polarization of the photon.

d) Since the above diagram is the only one contributing to this process at this order in perturbation theory we know that the Ward-identity should hold. Use this to make an educated guess of how the momentum dependence of the second order photon propagator $\Pi_2^{\mu\nu}$ looks like. i.e. find a suitable function $f$ in the relation: $\Pi_2^{\mu\nu} \propto f(p)$.