

Regularization in Quantum Field Theory from the Causal Point of View

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Abstract

The causal approach to perturbative quantum field theory is presented in detail, which goes back to a seminal work by Henri Epstein and Vladimir Jurko Glaser in 1973. Causal perturbation theory is a mathematically rigorous approach to renormalization theory, which makes it possible to put the theoretical setup of perturbative quantum field theory on a sound mathematical basis. Epstein and Glaser solved this problem for a special class of distributions, the time-ordered products, that fulfill a causality condition, which itself is a basic requirement in axiomatic quantum field theory. In their original work, Epstein and Glaser studied only theories involving scalar particles. In this review, the extension of the method to theories with higher spin, including gravity, is presented. Furthermore, specific examples are presented in order to highlight the technical differences between the causal method and other regularization methods, like, e.g. dimensional regularization.

Contents

1	Introduction	2
2	Mathematical Preliminaries	4
2.1	<i>Regularization of Distributions</i>	4
2.2	<i>Scaling Properties of Distributions</i>	7
2.3	<i>Splitting of Distributions</i>	9
3	Perturbative S-Matrix Theory	14
3.1	<i>Free Fields</i>	14
3.1.1	<i>Scalar Fields</i>	14
3.1.2	<i>Spin-1/2 Fields</i>	15
3.1.3	<i>Vector Fields</i>	17
3.1.4	<i>Spin-2 Fields</i>	21
3.2	<i>The Causal Structure of the Perturbative S-Matrix</i>	22

3.2.1	<i>General Construction of the S-Matrix</i>	22
3.2.2	<i>Example</i>	24
4	Regularization Methods	26
4.1	<i>Basic remarks</i>	26
4.2	<i>Scalar One-Loop Diagram in n Dimensions</i>	27
5	Comparison of Regularization Methods to the Causal Approach: Specific Examples	29
5.1	<i>Axial Anomalies</i>	29
5.2	<i>Schwinger Model</i>	35
5.2.1	<i>The Causal Approach</i>	35
5.2.2	<i>Dimensional Regularization</i>	37
5.3	<i>Scalar QED in 2+1 Dimensions</i>	39
5.3.1	<i>Causal Approach</i>	39
5.3.2	<i>Dimensional Regularization</i>	42
5.3.3	<i>Pauli-Villars Regularization</i>	44
6	Infrared Divergences	46
6.1	<i>Bremsstrahlung</i>	49
6.2	<i>Self Energy</i>	53
6.3	<i>Vertex Function</i>	55
7	Gauge Theories	58
7.1	<i>Spin 1</i>	58
7.2	<i>Spin 2</i>	66
8	Conclusion	70
A	Special Distributions in 3+1-Dimensional Spacetime	72

1 Introduction

Quantum field theory (QFT) is more singular than quantum mechanics. The basic mathematical objects of quantum mechanics are square integrable functions, whereas the corresponding central objects in QFT are generalized functions or distributions. A potential drawback of the theory of distributions for physics is the fact that it is a purely linear theory, in the sense that the product of two distributions cannot consistently be defined in general, as has been proved by Laurent Schwartz [1], who was awarded the Fields medal for his work on distributions in 1950.

If one is careless about this point the well-known ultraviolet (UV) divergences appear in perturbative quantum field theory (pQFT). The occurrence of these divergences is sometimes ascribed in a qualitative manner to problematic contributions of virtual particles with "very high energy", or, equivalently, to physical phenomena at very short spacetime distances, and put forward as an argument that the quantized version of extended objects like strings which are less singular than point-like particles should be used instead in QFT. In view of the fact that UV divergences can be circumvented by a proper treatment of distributions in pQFT, this argument for string theories is no longer convincing.

We illustrate the problem mentioned above by a naive example of a "UV divergence" by considering Heaviside- Θ - and Dirac- δ -distributions in 1-dimensional "configuration space". The product of these two

distributions $\Theta(x)\delta(x)$ is obviously ill-defined, however, the distributional Fourier transforms

$$\sqrt{2\pi}\mathcal{F}\{\delta\}(k) = \sqrt{2\pi}\hat{\delta}(k) = \int dx \delta(x)e^{-ikx} = 1, \quad (1)$$

$$\sqrt{2\pi}\hat{\Theta}(k) = \lim_{\epsilon \searrow 0} \int dx \Theta(x)e^{-ikx-\epsilon x} = \lim_{\epsilon \searrow 0} \frac{ie^{-ikx-\epsilon x}}{k-i\epsilon} \Big|_0^\infty = -\frac{i}{k-i0}, \quad (2)$$

exist and one may attempt to calculate the ill-defined product in "momentum space", which *formally* goes over into a convolution

$$\mathcal{F}\{\Theta\delta\}(k) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx}\Theta(x)\delta(x) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx} \int \frac{dk'}{\sqrt{2\pi}} \hat{\Theta}(k')e^{+ik'x} \int \frac{dk''}{\sqrt{2\pi}} \hat{\delta}(k'')e^{+ik''x}. \quad (3)$$

Throughout this paper, we use the symmetric definition of the (inverse) Fourier transform according to Eq. (7) and Eq. (8). Since $\int dx e^{i(k'+k''-k)x} = 2\pi\delta(k'+k''-k)$, we obtain

$$\mathcal{F}\{\Theta\delta\}(k) = \frac{1}{\sqrt{2\pi}} \int dk' \hat{\Theta}(k')\hat{\delta}(k-k') = -\frac{i}{(2\pi)^{3/2}} \int \frac{dk'}{k'-i0}. \quad (4)$$

The obvious problem in x-space leads to a "logarithmic UV divergence" in k-space. It will become clear below that a concise description of the scaling properties of distributions, related to the wide-spread notion of the superficial degree of divergence of Feynman integrals, is crucial for the correct treatment of singular products of distributions in pQFT.

In pQFT, the rôle of the Heaviside Θ -distribution is taken over by the time-ordering operator. The well-known textbook expression for the perturbative scattering matrix given by

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \dots \int_{-\infty}^{+\infty} dt_n T[H_{int}(t_1) \dots H_{int}(t_n)] \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots \int d^4x_n T[\mathcal{H}_{int}(x_1) \dots \mathcal{H}_{int}(x_n)], \end{aligned} \quad (5)$$

where the interaction Hamiltonian $H_{int}(t)$ is given by the interaction Hamiltonian density $\mathcal{H}_{int}(x)$ via $H_{int}(t) = \int d^3x \mathcal{H}_{int}(x)$, is problematic in the UV regime (and in the infrared regime, when massless fields are involved). A time-ordered expression à la

$$T[\mathcal{H}_{int}(x_1) \dots \mathcal{H}_{int}(x_n)] = \sum_{\text{Perm. } \Pi} \Theta(x_{\Pi_1}^0 - x_{\Pi_2}^0) \dots \Theta(x_{\Pi_{(n-1)}}^0 - x_{\Pi_n}^0) \mathcal{H}_{int}(x_{\Pi_1}) \dots \mathcal{H}_{int}(x_{\Pi_n}) \quad (6)$$

is formal (i.e., ill-defined), since the operator-valued distribution products of the \mathcal{H}_{int} are simply too singular to be multiplied by Θ -distributions.

In this review, the construction of pQFT is reviewed from a causal point of view with a special focus on the regularization of distributions. Typical examples are discussed in the causal framework and compared to the corresponding treatment in the Pauli-Villars regularization or dimensional regularization. In the last section we describe a modern approach to quantum gauge theories including gravity. This shows that the gauge principle in a suitable formulation is a universal principle of nature because it determines all interactions. Therefore, any regularization method must be in accordance with it.

2 Mathematical Preliminaries

2.1 Regularization of Distributions

Distributions are continuous linear functionals on certain function spaces. There exist different spaces of distributions. For quantum field theory the most important function space is the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. It consists of infinitely differentiable complex-valued functions of rapid decrease, that means the functions together with their derivatives fall off more quickly than the inverse of any polynomial. The reason for the importance of $\mathcal{S}(\mathbb{R}^n)$ is the fact that the Fourier transform (the expression px denotes a generalized n -dimensional Euclidean or Minkowski scalar product depending on the respective situation)

$$\mathcal{F}\{f\}(p) = \hat{f}(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ipx} d^n x, \quad (7)$$

is a linear bi-continuous bijection from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. Indeed, the inverse Fourier transform is given by

$$\mathcal{F}^{-1}\{g\}(x) = \check{g}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(p) e^{ipx} d^n p. \quad (8)$$

The dual space of $\mathcal{S}(\mathbb{R}^n)$ denoted by $\mathcal{S}'(\mathbb{R}^n)$, is the space of tempered distributions. A tempered distribution $d(f)$ is a continuous linear complex-valued functional on $\mathcal{S}(\mathbb{R}^n) \ni f$; we also write $d(f) = \langle d, f \rangle$. The functions $f \in \mathcal{S}(\mathbb{R}^n)$ are called test functions. The Fourier transform of a tempered distribution d is now simply defined by its action on the test functions:

$$\langle \mathcal{F}\{d\}, f \rangle \stackrel{\text{def}}{=} \langle d, \mathcal{F}\{f\} \rangle, \quad f \in \mathcal{S}(\mathbb{R}^n). \quad (9)$$

In this way, by operating on test functions, various operations on distributions like differentiation, convolution etc are defined. Note that the definition Eq. (9), $\hat{d}(f) = d(\hat{f})$, is sometimes written in an intuitive manner by the help of formal integrals

$$\int_{\mathbb{R}^n} \hat{d}(q) f(q) d^n q = \int_{\mathbb{R}^n} d(q) \hat{f}(q) d^n q, \quad (10)$$

showing the close relation of the definition above to the Plancherel theorem. Of course, the common physical distinction whether the integration variable q is in "real space" or "momentum space" is of no relevance here. For mathematical details concerning the properties of distributions, we refer to [2, 3].

The most important distributions for field theory are related to linear partial differential equations, for example the Klein-Gordon equation

$$(\square + m^2)d(x) = \left(\frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} + m^2 \right) d(x) = 0. \quad (11)$$

An important *distributional* solution of the 4-dimensional Klein-Gordon equation is the Jordan-Pauli distribution

$$D_m(x) = \frac{i}{(2\pi)^3} \int d^4 p \delta(p^2 - m^2) \text{sgn}(p_0) e^{-ipx} \quad (12)$$

where the integral must be understood as a distributional Fourier transform; the factor i makes $D_m(x)$ real. If we decompose the sign-function, $\text{sgn } p_0 = \Theta(p_0) - \Theta(-p_0)$, we obtain the decomposition of $D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x)$ into positive and negative frequency parts, for example

$$D_m^{(+)}(x) = \frac{i}{(2\pi)^3} \int d^4 p \delta(p^2 - m^2) \Theta(-p_0) e^{ipx} = \frac{i}{(2\pi)^3} \int d^4 p \delta(p^2 - m^2) \Theta(p_0) e^{-ipx} \quad (13)$$

In addition to these solutions of the homogeneous Klein-Gordon equation we need weak solutions of the inhomogeneous equation

$$(\square + m^2)D_{ret}^m(x) = \delta(x). \quad (14)$$

This retarded distribution which vanishes for negative time $ct = x_0 < 0$ (c is the velocity of light) is given by

$$D_{ret}^m(x) = \Theta(x_0)D^m(x) \quad (15)$$

and the corresponding advanced distribution by $D_{av}^m(x) = D_{ret}^m(-x)$. Finally, the so-called Feynman propagator is defined as

$$D_F^m(x) = D_{ret}^m(x) - D_{ret}^{m(-)}(x) = D_{av}^m(x) + D_m^{(+)}(x). \quad (16)$$

Its Fourier transform is equal to

$$D_F^m(x) = -(2\pi)^{-4} \int d^4p \frac{e^{-ipx}}{p^2 - m^2 + i0}, \quad (17)$$

where the symbol $i0$ stands for $i\varepsilon$ and the limit $\varepsilon \rightarrow 0$ in the distributional sense, i.e. in $\mathcal{S}'(\mathbb{R}^n)$.

In the appendix, a concise list of the fundamental free field commutators and propagators is given, where we also explicitly account for the most common conventions concerning the signs and normalizations of the distributions.

In standard QFT the Feynman propagator D_F^m is associated with the inner lines of a Feynman graph in the simplest case of scalar particles (spin 0). In a lowest-order loop graph there arises the problem of multiplying two Feynman propagators $D_F^{m_1}(x) \cdot D_F^{m_2}(x)$, a product which is ill-defined. In fact, in momentum space this product corresponds to a formal convolution of the form

$$\Sigma(p) = C \int d^4k D_{m_1}(k) D_{m_2}(p - k), \quad (18)$$

where C is a numerical constant; we shall always use the symbol C for constants which we do not compute explicitly because they are not interesting for our purpose. To simplify the notation the various C 's stand for different constants. By counting powers of $|k|$ we see that the integral Eq. (18) is logarithmically divergent in the ultraviolet regime $|k| \rightarrow \infty$. To make it well defined we use a regularization of the Feynman propagator $D_F^m(x)$

$$D_F^{\text{reg}}(k) = C \left(\frac{1}{k^2 - m^2 + i0} - \frac{1}{k^2 - M^2 + i0} \right) = C \frac{m^2 - M^2}{(k^2 - m^2 + i0)(k^2 - M^2 + i0)}, \quad (19)$$

where C denotes a real normalization constant which depends on specifically chosen conventions. Modifying the Feynman propagator according to Eq. (19) at a high mass or energy scale given M is the basic essence of the so-called Pauli-Villars regularization. Note that the propagator term containing M has the "wrong sign" and does not correspond to the contribution of a heavy physical particle. However, for $M \rightarrow \infty$, $D_F^{\text{reg}}(k)$ converges to $D_F^m(k)$ in the sense of tempered distributions. We present here one possible approach to calculate the scalar self-energy diagram. Using the Fourier transform

$$\frac{1}{k^2 - m^2 + i0} = \frac{1}{i} \int_0^\infty ds e^{is(k^2 - m^2 + i0)} \quad (20)$$

the regularized propagator is equal to

$$D_F^{\text{reg}}(k) = C \int_0^\infty ds e^{isk^2 - s0} \left(e^{-ism^2} - e^{-isM^2} \right). \quad (21)$$

Substituting the Feynman propagators in the self-energy integral Eq. (18) by regularized ones, we obtain a finite integral

$$\Sigma^{\text{reg}}(p) = C \int d^4k \int_0^\infty ds_1 \int_0^\infty ds_2 e^{is_1 k^2 - s_1 0} \left(e^{-is_1 m_1^2} - e^{-is_1 M^2} \right) \times \quad (22)$$

$$\times e^{is_2 (p-k)^2 - s_2 0} \left(e^{-is_2 m_2^2} - e^{-is_2 M^2} \right). \quad (23)$$

Here the 4-dimensional k -integral can be carried out by means of the Gauss-Fresnel integral

$$\int e^{i(ak^2 + bk)} d^4k = \frac{\pi^2}{ia^2} \exp\left(-\frac{ib^2}{4a}\right), \quad a > 0. \quad (24)$$

The result is

$$\Sigma^{\text{reg}}(p) = C \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{e^{-(s_1 + s_2)0}}{(s_1 + s_2)^2} \exp\left(\frac{is_1 s_2 s_1 + s_2^2}{p}\right) \left(e^{-is_1 m_1^2} - e^{-is_1 M^2} \right) \quad (25)$$

$$\times \left(e^{-is_2 m_2^2} - e^{-is_2 M^2} \right). \quad (26)$$

Now we introduce the new integration variables $t_1 = s_1 / (s_1 + s_2)$ and $t_2 = s_1 + s_2$, then we have

$$\Sigma_\varepsilon^{\text{reg}}(p) = C \int_0^1 dt_1 \int_0^\infty dt_2 \frac{1}{t_2^2} e^{-\varepsilon t_2 + it_1 t_2 (1-t_1) p^2} \quad (27)$$

$$\left(e^{-it_1 t_2 m_1^2} - e^{-it_1 t_2 M^2} \right) \left(e^{-i(1-t_1) t_2 m_2^2} - e^{-i(1-t_1) t_2 M^2} \right). \quad (28)$$

We have written $i\varepsilon$ for $i0$ and take the limit $\varepsilon \rightarrow 0$ later on. To perform the t_2 -integration we need the integral

$$\int_a^\infty \frac{dx}{x^2} e^{-\varepsilon x + izx} \stackrel{\text{def}}{=} J_a(z) \quad (29)$$

where we the lower limit of integration is $a > 0$ in order to avoid the singularity at $x = 0$; note that Eq. (28) is integrable at $t_2 = 0$. By differentiating twice with respect to z the denominator x^2 is removed and the integral can be easily evaluated

$$J_a''(z) = \frac{e^{iza - \varepsilon a}}{iz - \varepsilon}. \quad (30)$$

Now the limit $a \rightarrow 0$ is possible and two integrations in z yield

$$J_0(z) = -iz[\log(iz - \varepsilon) - 1]. \quad (31)$$

Using this result in Eq. (28) the regularized self-energy integral becomes

$$\Sigma_\varepsilon^{\text{reg}}(p) = C \int_0^1 dt_1 \left[z_1 \log(iz_1 - \varepsilon) - z_2 \log(iz_2 - \varepsilon) - z_3 \log(iz_3 - \varepsilon) + z_4 \log(iz_4 - \varepsilon) \right], \quad (32)$$

where

$$z_1 = t_1(1-t_1)p^2 - t_1 m_1^2 - (1-t_1)m_2^2 \quad (33)$$

$$z_2 = t_1(1-t_1)p^2 - t_1 m_1^2 - (1-t_1)M^2 \quad (34)$$

$$z_3 = t_1(1 - t_1)p^2 - t_1M^2 - (1 - t_1)m_2^2 \quad (35)$$

$$z_4 = t_1(1 - t_1)p^2 - t_1M^2 - (1 - t_1)M^2. \quad (36)$$

The integral Eq. (32) still diverges for $M \rightarrow \infty$. We have to split off the divergent part. This process, called renormalization, must always be combined with regularization. In order to obtain a unique finite result we proceed as follows. We compute the special value $\Sigma_\varepsilon^{\text{reg}}(0)$ and subtract it from (16). Then the limit

$$\lim_{\varepsilon \rightarrow 0} \lim_{M \rightarrow \infty} (\Sigma_\varepsilon^{\text{reg}}(p) - \Sigma_\varepsilon^{\text{reg}}(0)) \stackrel{\text{def}}{=} \Sigma'(p) \quad (37)$$

is finite. It satisfies the normalization condition

$$\Sigma'(0) = 0. \quad (38)$$

We will not calculate the finite self-energy $\Sigma'(p)$ explicitly because later on we shall discuss methods which give the result in a more elegant way. The subtraction of a constant in Eq. (37) is equivalent to the subtraction of a local term $\sim \delta(x)$ in x -space. If we considered a more singular distribution, then a certain polynomial in p must be subtracted which corresponds to a sum of derivatives of the δ -distribution in x -space.

2.2 Scaling Properties of Distributions

From the last section it is clear that the singular behavior of a tempered distribution $\hat{d}(p)$ at infinity or of its (inverse) Fourier transform $d(x)$ at $x = 0$ is of central importance in QFT. To study these properties the so-called quasi-asymptotics of a tempered distribution is very useful. The definition is the following:

Definition 2.1. The distribution $d(x) \in \mathcal{S}'(\mathbb{R}^m)$ has a quasi-asymptotics $d_0(x)$ at $x = 0$ with respect to a positive continuous function $\rho(\delta)$, $\delta > 0$, if the limit

$$\lim_{\delta \rightarrow 0} \rho(\delta) \delta^m d(\delta x) = d_0(x) \neq 0 \quad (39)$$

exists in $\mathcal{S}'(\mathbb{R}^m)$.

In the smeared out form of Eq. (39) with a test function $\varphi \in \mathcal{S}'(\mathbb{R}^m)$

$$\lim_{\delta \rightarrow 0} \rho(\delta) \left\langle d(x), \varphi\left(\frac{x}{\delta}\right) \right\rangle = \langle d_0, \varphi \rangle. \quad (40)$$

we go over to momentum space to find an equivalent condition for the Fourier transform $\hat{d}(p)$. Since

$$\left\langle d(x), \varphi\left(\frac{x}{\delta}\right) \right\rangle = \left\langle \hat{d}(p), \left(\varphi\left(\frac{x}{\delta}\right)\right)^\sim(p) \right\rangle = \delta^m \langle \hat{d}(p), \check{\varphi}(\delta p) \rangle = \left\langle \hat{d}\left(\frac{p}{\delta}\right), \check{\varphi}(p) \right\rangle, \quad (41)$$

where $\check{\varphi}$ denotes the inverse Fourier transform, we get the following equivalent definition:

Definition 2.2. The distribution $\hat{d}(p) \in \mathcal{S}'(\mathbb{R}^m)$ has quasi-asymptotics $\hat{d}_0(p)$ at $p = \infty$ if

$$\lim_{\delta \rightarrow 0} \rho(\delta) \left\langle \hat{d}\left(\frac{p}{\delta}\right), \check{\varphi}(p) \right\rangle = \langle \hat{d}_0, \check{\varphi} \rangle \quad (42)$$

exists for all $\check{\varphi} \in \mathcal{S}(\mathbb{R}^m)$.

In momentum space the quasi-asymptotics controls the ultraviolet behavior of the distribution. Let us consider a scaling transformation

$$\lim_{\delta \rightarrow 0} \rho(\delta) \langle \hat{d}\left(\frac{p}{\delta}\right), \check{\varphi}(ap) \rangle = \langle \hat{d}_0(p), \check{\varphi}(ap) \rangle \quad (43)$$

$$= a^{-m} \lim_{\delta \rightarrow 0} \rho(\delta) \left\langle \hat{d}\left(\frac{p}{a\delta}\right), \check{\varphi}(p) \right\rangle = a^{-m} \lim_{\delta \rightarrow 0} \frac{\rho(\delta)}{\rho(a\delta)} \rho(a\delta) \left\langle \hat{d}\left(\frac{p}{a\delta}\right), \check{\varphi}(p) \right\rangle. \quad (44)$$

Since

$$\lim_{\delta \rightarrow 0} \rho(a\delta) \langle \hat{d}(\frac{p}{a\delta}), \check{\varphi}(p) \rangle = \langle \hat{d}_0(p), \check{\varphi}(p) \rangle \quad (45)$$

exists, we may conclude that the limit

$$\lim_{\delta \rightarrow 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^{-m} \frac{\langle \hat{d}_0(p), \check{\varphi}(p) \rangle}{\langle \hat{d}_0(p), \check{\varphi}(ap) \rangle} \stackrel{\text{def}}{=} \rho_0(a) \quad (46)$$

exists, too, assuming that the denominator is different from 0. By another scaling transformation it follows

$$\rho_0(ab) = \rho_0(a)\rho_0(b), \quad (47)$$

which implies $\rho_0(a) = a^\omega$ with some real ω . We therefore call $\rho(\delta)$ the power-counting function.

With help of the power-counting function we can now define the singular order of a distribution.

Definition 2.3. The distribution $d \in \mathcal{S}'(R^m)$ is called singular of order ω , if it has a quasi-asymptotics $d_0(x)$ at $x = 0$, or its Fourier transform has quasi-asymptotics $\hat{d}_0(p)$ at $p = \infty$, respectively, with power-counting function $\rho(\delta)$ satisfying

$$\lim_{\delta \rightarrow 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^\omega, \quad (48)$$

for each $a > 0$.

Eq. (46) implies

$$a^m \langle \hat{d}_0(p), \check{\varphi}(ap) \rangle = \langle \hat{d}_0(\frac{p}{a}), \check{\varphi}(p) \rangle = a^{-\omega} \langle \hat{d}_0(p), \check{\varphi}(p) \rangle \quad (49)$$

$$= \langle d_0(x), \varphi(\frac{x}{a}) \rangle = a^m \langle d_0(ax), \varphi(x) \rangle = a^{-\omega} \langle d_0(x), \varphi(x) \rangle, \quad (50)$$

i.e. \hat{d}_0 is homogeneous of degree ω :

$$\hat{d}_0(\frac{p}{a}) = a^{-\omega} \hat{d}_0(p), \quad (51)$$

$$d_0(ax) = a^{-(m+\omega)} d_0(x). \quad (52)$$

This implies that d_0 has quasi-asymptotics $\rho(\delta) = \delta^\omega$ and the singular order ω , too. A positive measurable function $\rho(\delta)$, satisfying Eq. (48), is called regularly varying at zero by mathematicians [4]. The power-counting function satisfies the following estimates: If $\varepsilon > 0$ is an arbitrarily small number, then there exist constants C, C' and δ_0 , such that

$$C\delta^{\omega+\varepsilon} \geq \rho(\delta) \geq C'\delta^{\omega-\varepsilon}, \quad (53)$$

for $\delta < \delta_0$.

We want to apply the definitions to the following examples:

- 1) $d = 1$: From Eq. (39) we get $\rho(\delta) = \delta^{-m}$ and $\omega = -m$.
- 2) $d(x) = D^a \delta(x)$ where

$$D^a \stackrel{\text{def}}{=} \frac{\partial^{a_1+\dots+a_m}}{\partial x_1^{a_1} \dots \partial x_m^{a_m}}, \quad |a| = a_1 + \dots + a_m. \quad (54)$$

Since

$$\hat{d}(p) = (2\pi)^{-m/2} (ip)^a, \quad (55)$$

we obtain $\rho(\delta) = \delta^{|a|}$ from Eq. (42) and $\omega = |a|$.

- 3) Let us consider the Jordan-Pauli distribution Eq. (12) which has the following form in x -space

$$D_m(x) = \frac{\text{sgn}x^0}{2\pi} \left[\delta(x^2) - \frac{m}{2} \frac{\Theta(x^2)}{\sqrt{x^2}} J_1(m\sqrt{x^2}) \right], \quad (56)$$

where J_1 is the Bessel function. We shall write the δ -distribution with argument always in contrast to the positive scaling factor δ . The one-dimensional δ -distribution satisfies

$$\delta(\delta^2 x^2) = \frac{\delta(x^2)}{\delta^2}, \quad (57)$$

whereas the term with the Bessel function stays bounded for $\delta\sqrt{x^2} \rightarrow 0$. Hence

$$\lim_{\delta \rightarrow 0} \delta^2 D_m(\delta x) = \frac{\text{sgn} x^0}{2\pi} \delta(x^2) = D_0(x) \quad (58)$$

which is just the mass zero Jordan-Pauli distribution. This illustrates the general fact that the quasi-asymptotics d_0 is given by the corresponding mass zero distribution. Since the Jordan-Pauli distribution is typically considered in \mathbb{R}^4 ($m = 4$), we find $\rho(\delta) = \delta^{-2}$ and $\omega(D_m) = -2$.

4) The positive frequency part Eq. (13)

$$\hat{D}_m^{(+)}(p) = \frac{i}{2\pi} \Theta(p^0) \delta(p^2 - m^2) \quad (59)$$

is best considered in momentum space. Since

$$\int \Theta\left(\frac{p_0}{\delta}\right) \delta\left(\frac{p^2}{\delta^2} - m^2\right) \varphi(p) d^4 p = \delta^2 \int \Theta(p_0) \delta(p^2 - \delta^2 m^2) \varphi(p) d^4 p \quad (60)$$

$$= \delta^2 \int \frac{d^3 p}{2\sqrt{p^2 + \delta^2 m^2}} \varphi(\sqrt{p^2 + \delta^2 m^2}, \vec{p}), \quad (61)$$

we find

$$\lim_{\delta \rightarrow 0} \delta^{-2} \hat{D}_m^{(+)}\left(\frac{p}{\delta}\right) = \hat{D}_0^{(+)}(p) \quad (62)$$

which implies $\omega(D_m^{(+)}) = -2$, in agreement with the foregoing example. We obviously have $\omega(D_m^{(-)}) = -2$, too.

We notice from example 2 that the degree of singularity at $x = 0$ increases with $\omega > 0$. The distributions with negative ω have only mild singularities. This difference will be important in the next section.

2.3 Splitting of Distributions

In QFT the problem arises of multiplying certain distributions which are singular at $x = 0$ by the discontinuous step function $\Theta(x_0)$, where x_0 is time. We will consider this problem only for distributions with a causal support: Let $d(x) = d(x_1, \dots, x_n) \in \mathcal{S}'(\mathbb{R}^m)$ where $x_j \in \mathbb{R}^4$, $m = 4n$, be a tempered distribution depending on n space-time arguments. By

$$\overline{V^+}(0) = \{x \mid x^2 = x_0^2 - \vec{x}^2 \geq 0, x_0 \geq 0\} \quad (63)$$

we denote the closed forward cone, and by

$$\overline{V^-}(0) = \{x \mid x^2 \geq 0, x_0 \leq 0\} \quad (64)$$

the closed backward cone. The n -dimensional generalizations are

$$\Gamma_n^\pm(0) = \{(x_1, \dots, x_n) \mid x_j \in \overline{V^\pm}(0), \forall j = 1, \dots, n\}. \quad (65)$$

The distribution $d(x)$ has causal support if

$$\text{supp } d \subseteq \Gamma_n^+(x) \cup \Gamma_n^-(0). \quad (66)$$

This means that all n space-time points are either in the forward light-cone or in the backward cone. The splitting problem now is to decompose such a distribution into a retarded minus advanced part

$$d(x) = r(x) - a(x), \quad (67)$$

where $\text{supp } r \subseteq \Gamma_n^+$ and $\text{supp } a \subseteq \Gamma_n^-$.

The simplest example of a causal distribution is the Jordan-Pauli distribution given by Eq. (56) where the splitting into retarded minus advanced distributions is trivially possible, see Eq. (15). It is misleading that the Feynman propagator $D_F^m(x)$ is also called "causal" sometimes, because it does not have a causal support due to the presence of $D_m^{(-)}$ in Eq. (16). In the general case we have to distinguish two cases:

a) Singular order $\omega < 0$: In this case, the power-counting function goes to infinity by Eq. (53)

$$\rho(\delta) \rightarrow \infty \quad \text{for } \delta \rightarrow 0. \quad (68)$$

This implies

$$\left\langle d(x), \varphi\left(\frac{x}{\delta}\right) \right\rangle \rightarrow \frac{\langle d_0, \varphi \rangle}{\rho(\delta)} \rightarrow 0. \quad (69)$$

We choose a monotonous C^∞ -function χ_0 over \mathbb{R}^1 with

$$\chi_0(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ < 1 & \text{for } 0 < t < 1 \\ 1 & \text{for } t \geq 1. \end{cases} \quad (70)$$

In addition we choose a vector $v = (v_1, \dots, v_{n-1}) \in \Gamma^+$, which means that all four-vectors v_j are inside the forward cone V^+ . Then

$$v \cdot x = \sum_{j=1}^{n-1} v_j \cdot x_j = 0 \quad (71)$$

is a space-like hyperplane that separates the causal support: All products $v_j \cdot x_j$ are either ≥ 0 for $x \in \Gamma^+$ or ≤ 0 for $x \in \Gamma^-$. Then as a consequence of Eq. (69) the limit

$$\lim_{\delta \rightarrow 0} \chi_0\left(\frac{v \cdot x}{\delta}\right) d(x) \stackrel{\text{def}}{=} \Theta(v \cdot x) d(x) = r(x) \quad (72)$$

exists. This is the case of trivial splitting where the multiplication by step function is possible. The result is independent of v .

b) $\omega \geq 0$: Now the power-counting function satisfies

$$\frac{\rho(\delta)}{\delta^{\omega+1}} \rightarrow \infty \quad \text{for } \delta \rightarrow 0. \quad (73)$$

To get a vanishing scaling limit as in Eq. (69) we choose a multi-index b with $|b| = \omega + 1$ and consider

$$\langle d(x) x^b, \psi\left(\frac{x}{\delta}\right) \rangle = \langle d(\delta y) y^b, \psi(y) \rangle \delta^{m+\omega+1} \rightarrow \langle d_0(y), y^b \psi \rangle \frac{\delta^{\omega+1}}{\rho(\delta)} \rightarrow 0. \quad (74)$$

It follows that the splitting as in case a) is possible if the test function φ satisfies

$$D^a \varphi(0) = 0 \quad \text{for } |a| \leq \omega. \quad (75)$$

To achieve that, we introduce an auxiliary function $w(x) \in \mathcal{S}(\mathbb{R}^m)$ with

$$w(0) = 1, \quad D^a w(0) = 0 \quad \text{for } 1 \leq |a| \leq \omega, \quad (76)$$

and define

$$(W\varphi)(x) \stackrel{\text{def}}{=} \varphi(x) - w(x) \sum_{|a|=0}^{\omega} \frac{x^a}{a!} (D^a \varphi)(0) = \sum_{|b|=\omega+1} x^b \psi_b(x). \quad (77)$$

The function $w(x)$ serves for the purpose of getting rapid decrease for $|x| \rightarrow \infty$. Now the decomposition according to a) Eq. (72) is possible

$$\langle r(x), \varphi \rangle \stackrel{\text{def}}{=} \langle \Theta(v \cdot x) d, W\varphi \rangle, \quad (78)$$

$$a(x) = r - d. \quad (79)$$

After construction $r(x)$ defines a tempered distribution with $\text{supp } r \subseteq \Gamma^+(0)$. It agrees with $d(x)$ on $\Gamma^+(0) \setminus \{0\}$ in the sense of distributions, because a test function $\varphi \in \mathcal{S}$ with $\text{supp } \varphi \subset \Gamma^+(0) \setminus \{0\}$ vanishes at $x = 0$, together with all its derivatives, so that the additional subtracted terms in Eq. (77) are 0. But without these terms, there is no splitting of $d(x)$ which makes sense for arbitrary $\varphi \in \mathcal{S}$, because the limit Eq. (72) exists on subtracted test functions only. *If one does the splitting incorrectly by simple multiplication with $\Theta(v \cdot x)$ as in a), one is punished by the well-known ultraviolet divergences in quantum field theory.* As we will discuss in detail later, these divergences appear in loop graphs which have $\omega \geq 0$. For those graphs the naive splitting with $\Theta(v \cdot x)$ is impossible and, as a consequence, the Feynman rules do not hold.

Again we have

$$\omega(r) = \omega(d) = \omega(a), \quad (80)$$

This is a direct consequence of the definitions Eq. (77) and Eq. (78), because the limit

$$\lim_{\delta \rightarrow 0} \rho(\delta) \left\langle r(x), \varphi\left(\frac{x}{\delta}\right) \right\rangle = \lim_{\delta \rightarrow 0} \rho(\delta) \left\langle d(x), \Theta W\left(\varphi\left(\frac{x}{\delta}\right)\right) \right\rangle \quad (81)$$

$$= \lim_{\delta \rightarrow 0} \rho(\delta) \langle d(x), (\Theta W\varphi)\left(\frac{x}{\delta}\right) \rangle = \langle d_0(x), (\Theta W\varphi)(x) \rangle \quad (82)$$

exists with the same power counting function as $d(x)$. But in sharp contrast to case a), the splitting b) is not unique. If $\tilde{r}(x)$ is the retarded part of another decomposition, then the difference

$$\tilde{r} - r = \sum_{|a|=0}^{\omega} \tilde{C}_a D^a \delta(x) \quad (83)$$

is again a distribution with point support. Since $\omega > 0$, this time the splitting is only determined up to a finite sum of local terms according to Eq. (83). These undetermined local terms are not fixed by causality, additional physical normalization conditions are necessary to fix them.

Before we proceed, it might help to provide some intuitive understanding of the distribution splitting process. One should remember the fact that the distributions appearing in local quantum field theory are more singular than ordinary functions, such that the products of the distributions are not necessarily well defined *ab initio*. E.g., the Feynman propagator can be calculated in configuration space [5]

$$D_F^m(x) = \frac{1}{4\pi} \delta(x^2) - \frac{m}{8\pi\sqrt{x^2}} \Theta(x^2) [J_1(m\sqrt{x^2}) - iN_1(m\sqrt{x^2})] + \frac{im}{(2\pi)^2\sqrt{-x^2}} \Theta(-x^2) K_1(m\sqrt{-x^2}), \quad (84)$$

where J_1 , N_1 , and K_1 are Bessel functions. For $x^2 \sim 0$, Eq. (84) can be decomposed according to

$$D_F^m(x) = \frac{1}{(2\pi)^2} \frac{1}{x^2 - i0} + O(m^2 x^2) = D_F^0(x) + O(m^2 x^2). \quad (85)$$

A formal product like $D_F^m(x) D_F^m(x)$ contains the highly singular (formal) expression $1/(x^2 - i0)^2 \sim 1/x^4$, and it is not trivial to understand the precise meaning of such a singular expression in the vicinity of the

light-cone where $x^2 \sim 0$. In general, the definition of distributional products works better in momentum space, where the analytic behavior of distributions appearing in quantum field theory is smoother and where the product goes over into a convolution. Still, the true difficulty is located in the point $x = 0$, where the distributional behavior of the product of two Feynman propagators is no longer mathematically meaningful. In momentum space, this problem leads to a logarithmically divergent integral. This is the point where the causal method provides the well-defined tools to isolate this ill-defined part of the product from the regular part on $\mathbb{R}^4 \setminus \{0\}$. Generally, perturbation theory itself is unable to describe local, "zero-distance" interactions without further input. At least, if the mathematics is done correctly and distributions are treated correctly, then all results remain finite at every calculational step, i.e. well-defined. The point $x = 0$ is essential because the distributions are most singular there. The subtracted terms have no direct physical meaning because they remain with free parameters, this is the freedom of (finite) renormalization, which is discussed in further detail in the sequel.

For practical reasons, explicit calculations in quantum field theory are usually done in momentum space. As a natural consequence, we must investigate the splitting procedure in p -space. We need the distributional Fourier transforms

$$\mathcal{F}^{-1}[\Theta(v \cdot x)] \stackrel{\text{def}}{=} \check{\chi}(k) \quad (86)$$

$$\mathcal{F}^{-1}[x^a w](p) = (iD_p)^a \check{w}(p). \quad (87)$$

Since

$$(D^a \varphi)(0) = \langle (-)^a D^a \delta, \varphi \rangle = (-)^a \langle \widehat{D^a \delta}, \check{\varphi} \rangle \quad (88)$$

$$= (-)^a (2\pi)^{-m/2} \langle (-ip)^a, \check{\varphi} \rangle = (2\pi)^{-m/2} \langle (ip)^a, \check{\varphi} \rangle, \quad (89)$$

we conclude from Eq. (78) that

$$\langle \hat{r}, \check{\varphi} \rangle = \langle \hat{d}, (\Theta W \varphi)^\check{\sim} \rangle = (2\pi)^{-m/2} \left\langle \hat{d}, \check{\chi} * \left[\check{\varphi} - \sum_{|a|=0}^{\omega} \frac{1}{a!} (iD_p)^a \check{w}(p) (2\pi)^{-m/2} \langle (ip')^a, \check{\varphi} \rangle \right] \right\rangle_p \quad (90)$$

$$= (2\pi)^{-m/2} \left\langle \hat{\chi} * \hat{d}, \check{\varphi} - \sum_{|a|=0}^{\omega} \dots \right\rangle, \quad (91)$$

where the asterisk means convolution. We stress the fact that the convolution $\hat{\chi} * \hat{d}$ is only defined on subtracted test functions, not on $\check{\varphi}$ alone. Interchanging p' and p in the subtraction terms, we may write

$$\langle \hat{r}, \check{\varphi} \rangle = (2\pi)^{-m/2} \int dk \hat{\chi}(k) \left\langle \hat{d}(p-k) - (2\pi)^{-m/2} \sum_a \frac{(-)^a}{a!} p^a \int dp' \hat{d}(p'-k) D_{p'}^a \check{w}(p'), \check{\varphi} \right\rangle_p. \quad (92)$$

After partial integration in the p' -integral this is equivalent to the following result for the retarded distribution

$$\hat{r}(p) = (2\pi)^{-m/2} \int dk \hat{\chi}(k) \left[\hat{d}(p-k) - (2\pi)^{-m/2} \sum_{|a|=0}^{\omega} \frac{p^a}{a!} \int dp' (D_{p'}^a \hat{d}(p'-k)) \check{w}(p') \right]. \quad (93)$$

Here the k -integral is understood in the sense of distributions as in (59).

By considering the Fourier transform of Eq. (83) we see that $\hat{r}(p)$ is only determined up to a polynomial in p of degree ω . Consequently the general result for the retarded distribution reads

$$\hat{\hat{r}}(p) = \hat{r}(p) + \sum_{|a|=0}^{\omega} C_a p^a \quad (94)$$

with $\hat{r}(p)$ given by Eq. (93). We now assume that there exists a point $q \in \mathbb{R}^m$ where the derivatives $D^b \hat{r}(q)$ exist in the usual sense of functions for all $|b| \leq \omega$. Let us define

$$\hat{r}_q(p) = \hat{r}(p) - \sum_{|b|=0}^{\omega} \frac{(p-q)^b}{b!} D^b \hat{r}(q). \quad (95)$$

This is another retarded distribution because we have only added a polynomial in p of degree ω . Furthermore, this solution of the splitting problem is *uniquely* specified by the normalization condition

$$D^b \hat{r}_q(q) = 0, \quad |b| \leq \omega. \quad (96)$$

We compute

$$D^b \hat{r}(q) = (2\pi)^{-m/2} \int dk \hat{\chi}(k) \left[(D^b \hat{d})(q-k) \right] \quad (97)$$

$$- (2\pi)^{-m/2} \sum_{b \leq a} \frac{a! q^{a-b}}{(a-b)! a!} \int dp' \check{w}(p') D_p^a \hat{d}(p'-k) \quad (98)$$

from Eq. (93) and substitute this into Eq. (95). Since

$$\sum_{b \leq a} \frac{(p-q)^b}{b!} \frac{q^{a-b}}{(a-b)!} = \frac{1}{a!} \sum_{b \leq a} \binom{a}{b} (p-q)^b q^{a-b} = \frac{p^a}{a!}, \quad (99)$$

the subtracted terms in Eq. (93) drop out

$$\hat{r}_q(p) = (2\pi)^{-m/2} \int dk \hat{\chi}(k) \left[\hat{d}(p-k) - \sum_{|b|=0}^{\omega} \frac{(p-q)^b}{b!} (D^b \hat{d})(q-k) \right]. \quad (100)$$

This is the splitting solution with normalization point q . It is uniquely specified by Eq. (96), that means it does not depend on the time-like vector v in Eq. (86). The subtracted terms are the beginning of the Taylor series at $p = q$. This is an ultraviolet "regularization" in the usual terminology. It should be stressed, however, that here this is a consequence of the causal distribution splitting and not an ad hoc recipe.

It is well-known that causality can be expressed in momentum space by dispersion relations. Therefore we look for a connection of the result Eq. (100) with dispersion relations. We take $q = 0$ in Eq. (100), which is possible if all fields are massive, for example, and consider time-like $p \in \Gamma^+$. We choose a special coordinate system such that $p = (p_1^0, \vec{0}, 0, \dots)$. Note that this coordinate system is not obtained by a Lorentz transformation from the original one, but by an orthogonal transformation in \mathbb{R}^m . Furthermore we take v parallel to p , i.e. $v = (1, \vec{0}, 0, \dots)$. Then v varies with p , but this is admissible because Eq. (100) is actually independent of v . We now have $\Theta(v \cdot x) = \Theta(x_1^0)$ and the Fourier transform (54) is given by

$$\hat{\chi}(k) = (2\pi)^{m/2-1} \delta(\vec{k}_1, k_2, \dots, k_{n-1}) \frac{i}{k_1^0 + i0}. \quad (101)$$

We always use the mathematical notation $i0$ for $i\varepsilon$ with the subsequent distributional limit $\varepsilon \rightarrow 0$. Using this result in Eq. (100) we shall obtain

$$\hat{r}_0(p_1^0) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dk_1^0 \frac{1}{k_1^0 + i0} \left[\hat{d}(p_1^0 - k_1^0, 0, \dots) \right] \quad (102)$$

$$- \sum_{a=0}^{\omega} \frac{(p_1^0)^a}{a!} (-)^a D_{k_1^0}^a \hat{d}(q_1^0 - k_1^0, 0, \dots) \Big|_{q_1^0=0} \Big]. \quad (103)$$

The transformation of this result to the usual form of a dispersion integral leads to the following result:

$$r_0(p_1^0) = \frac{i}{2\pi} (p_1^0)^{\omega+1} \int_{-\infty}^{+\infty} dk_0 \frac{\hat{d}(k_0)}{(k_0 - i0)^{\omega+1} (p_1^0 - k_0 + i0)}. \quad (104)$$

The proof is given in [6], proposition 2.4.1. This expression is a subtracted dispersion relation. To write down the result for arbitrary $p \in \Gamma^+$, we use the variable of integration $t = k_0/p_1^0$ and arrive at

$$\hat{r}_0(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp)}{(t - i0)^{\omega+1} (1 - t + i0)}. \quad (105)$$

For later reference we call this the central splitting solution, because it is normalized at the origin ($q = 0$ in Eq. (96)). The latter fact has two important consequences. (i) The central splitting solution does not introduce a new mass scale into the theory. If $q \neq 0$, then $|q^2| = M^2$ defines such a scale. (ii) Most symmetry properties of $\hat{d}(p)$ are preserved under central splitting, as we will see later, because the origin $q = 0$ is a very symmetrical point. In the self-energy computation in sect. 2.1 by means of Pauli-Villars regularization we have also calculated this central solution Eq. (38).

It is easy to verify that the dispersion integral Eq. (68) is convergent for $|t| \rightarrow \infty$. But it would be ultraviolet divergent, if ω in Eq. (105) is chosen too small. Consequently, *the correct distribution splitting with the right singular order ω is terribly important. Incorrect distribution splitting leads to ultraviolet divergences.* This is the origin of the ultraviolet problem in quantum field theory.

3 Perturbative S-Matrix Theory

In perturbation theory all quantities are expanded in terms of free fields. To decide which free fields are relevant we notice that all interactions in nature can be described by quantum gauge theories, *gravity included*. Therefore, it is important to discuss quantized free gauge fields and their gauge structure. The latter is defined by means of ghost fields. In contrast to the functional integral approach to QFT where the Faddeev-Popov ghosts play indeed a somewhat ghost-like rôle, these are genuine dynamical fields in the causal approach. Regarding regularization it is a subtle problem to perform it in a way such that gauge invariance is conserved. In this respect dimensional regularization is technically advantageous.

3.1 Free Fields

3.1.1 Scalar Fields

First we consider a neutral scalar field with mass m which is a solution of the Klein-Gordon equation

$$(\square + m^2)\varphi(x) = 0. \quad (106)$$

A real *classical* solution of this equation is given by

$$\varphi(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2E}} \left(a(\vec{p}) e^{-ipx} + a^*(\vec{p}) e^{ipx} \right), \quad (107)$$

where

$$px = p^0 x^0 - \vec{p} \cdot \vec{x} = p_\mu x^\mu, \quad E = +\sqrt{\vec{p}^2 + m^2} = p^0. \quad (108)$$

In quantum field theory $a(\vec{p})$ and $a^*(\vec{p})$ become operator-valued distributions satisfying the commutation relations

$$[a(\vec{p}), a^\dagger(\vec{q})] = \delta^{(3)}(\vec{p} - \vec{q}). \quad (109)$$

where $\delta^{(3)}$ denotes the Dirac's δ -distribution, all other commutators vanish. The quantized Bose field is now given by

$$\varphi(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2E}} \left(a(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx} \right). \quad (110)$$

The cross denotes the Hermitian conjugate. In fact, there exists a Fock-Hilbert space representation of the Bose field which proves the consistency of the quantization. φ is obviously Hermitian $\varphi^\dagger(x) = \varphi(x)$.

Let us call the second term in Eq. (110) involving a^\dagger the creation part $\varphi^{(+)}$ and the first term with $a(\vec{p})$ the absorption part $\varphi^{(-)}$. Then by Eq. (109) their commutator is equal to

$$[\varphi^{(-)}(x), \varphi^{(+)}(y)] = (2\pi)^{-3/2} \int \frac{d^3p}{2E} e^{-ip(x-y)} = -iD_m^{(+)}(x-y). \quad (111)$$

In the same way we get

$$[\varphi^{(+)}(x), \varphi^{(-)}(y)] = -iD_m^{(-)}(x-y). \quad (112)$$

Then the commutation relation for the total scalar field is given by the Jordan-Pauli distribution

$$[\varphi(x), \varphi(y)] = -iD_m(x-y). \quad (113)$$

The charged scalar field involves a slight generalization of the neutral one:

$$\varphi(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2E}} \left(a(\vec{p})e^{-ipx} + b^\dagger(\vec{p})e^{ipx} \right). \quad (114)$$

It contains two different kinds of particles whose absorption and emission operators satisfy

$$[a(\vec{p}), a^\dagger(\vec{q})] = \delta(\vec{p} - \vec{q}) = [b(\vec{p}), b^\dagger(\vec{q})] \quad (115)$$

and all other commutators vanish. Then it follows

$$[\varphi(x), \varphi(y)^\dagger] = -iD_m(x-y) \quad (116)$$

but $[\varphi(x), \varphi(y)] = 0$.

3.1.2 Spin-1/2 Fields

Spin-1/2 fields are needed to describe leptons and quarks. Spinor fields are solution of the Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) = m\psi(x). \quad (117)$$

The γ -matrices obey the anticommutation relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \quad (118)$$

To define the quantized Dirac field we consider a solution of Eq. (117) of the following form

$$\psi(x) = (2\pi)^{-3/2} \int d^3p \sum_{s=\pm 1} [u_s(\vec{p})e^{-ipx} b_s(\vec{p}) + v_s(\vec{p})e^{ipx} d_s^\dagger(\vec{p})], \quad (119)$$

$$\stackrel{\text{def}}{=} \psi^{(-)} + \psi^{(+)}. \quad (120)$$

The u - and v -spinors herein are obtained from the Fourier transformed equations

$$(p_\mu \gamma^\mu - m)u_s(\vec{p}) = 0 \quad (121)$$

$$(p_\mu \gamma^\mu + m)v_s(-\vec{p}) = 0, \quad (122)$$

with the normalization

$$u_s^\dagger(\vec{p})u_{s'}(\vec{p}) = \delta_{ss'} = v_s^\dagger(\vec{p})v_{s'}(\vec{p}) \quad (123)$$

$$u_s^\dagger(\vec{p})v_{s'}(-\vec{p}) = 0 = v_s^\dagger(-\vec{p})u_{s'}(\vec{p}) \quad (124)$$

$$u_s^\dagger(\vec{p})\gamma^0 u_{s'}(\vec{p}) = \frac{m}{E}\delta_{ss'} = -v_s(\vec{p})^\dagger \gamma^0 v_{s'}(\vec{p}). \quad (125)$$

The u - and v -spinors span the positive and negative spectral subspaces of the Dirac operator, respectively, which are defined by the projection operators

$$P_+(\vec{p}) = \sum_s u_s(\vec{p})u_s^\dagger(\vec{p}) = \left(\frac{\not{p} + m}{2E}\right)\gamma^0 \quad (126)$$

$$P_-(\vec{p}) = \sum_s v_s(-\vec{p})v_s^\dagger(-\vec{p}) = \left(\frac{\not{p} - m}{2E}\right)\gamma^0, \quad (127)$$

where

$$\not{p} = p_\mu \gamma^\mu, \quad p_0 = E = \sqrt{\vec{p}^2 + m^2}. \quad (128)$$

The projections are orthogonal

$$P_\pm(\vec{p})^2 = P_\pm(\vec{p}), \quad P_+(\vec{p}) + P_-(\vec{p}) = \mathbf{1}. \quad (129)$$

The quantization of the Dirac field is easily achieved by considering the b 's and d 's as operator-valued distributions satisfying the anticommutation relations

$$\{b_s(\vec{p}), b_{s'}^\dagger(\vec{q})\} = \delta_{ss'}\delta^{(3)}(\vec{p} - \vec{q}) = \{d_s(\vec{p}), d_{s'}^\dagger(\vec{q})\}, \quad (130)$$

and all other anticommutators vanish. We do not treat the Majorana case for neutral spin-1/2 fermions here. Then, the operators b and d can be interpreted as annihilation and their adjoints b^\dagger, d^\dagger as creation operators and the Fock space can be constructed from a unique vacuum in the usual way ([7], sect. 2.2). To get the anticommutation relations for the whole Dirac field we need the adjoint Dirac field

$$\psi^\dagger(x) = (2\pi)^{-3/2} \int d^3p [b_s^\dagger(\vec{p})u_s(\vec{p})^\dagger e^{ipx} + d_s(\vec{p})v_s(\vec{p})^\dagger e^{-ipx}]. \quad (131)$$

Multiplying by γ^0 , we get the so-called Dirac adjoint

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0 = \bar{\psi}^{(+)} + \bar{\psi}^{(-)} \quad (132)$$

$$\bar{\psi}^{(+)} = (2\pi)^{-3/2} \int d^3p b_s^\dagger(\vec{p})\bar{u}_s(\vec{p})e^{ipx} \quad (133)$$

$$\bar{\psi}^{(-)}(x) = (2\pi)^{-3/2} \int d^3p d_s(\vec{p})\bar{v}_s(\vec{p})e^{-ipx}. \quad (134)$$

With the aid of Eq. (130) we find

$$\{\psi_a^{(-)}(x), \bar{\psi}_b^{(+)}(y)\} = (2\pi)^{-3} \int d^3p u_{sa}(\vec{p})\bar{u}_{sb}(\vec{p})e^{-ip(x-y)}. \quad (135)$$

In the result Eq. (135), the covariant positive spectral projection operator Eq. (126) appears

$$\{\psi^{(-)}(x), \bar{\psi}^{(+)}(y)\} = (2\pi)^{-3} \int \frac{d^3p}{2E} (\not{p} + m) e^{-ip(x-y)} \stackrel{\text{def}}{=} \frac{1}{i} S_m^{(+)}(x-y) \quad (136)$$

$$= -i(i\not{\partial} + m) D_m^{(+)}(x-y). \quad (137)$$

In the same way, one obtains the other non-vanishing anticommutator

$$\{\psi^{(+)}(x), \bar{\psi}^{(-)}(y)\} \stackrel{\text{def}}{=} \frac{1}{i} S_m^{(-)}(x-y) = (2\pi)^{-3} \int \frac{d^3p}{2E} (\not{p} - m) e^{ip(x-y)} \quad (138)$$

$$= -i(i\not{\partial} + m) D_m^{(-)}(x-y). \quad (139)$$

This gives the anticommutation relation for the total Dirac field

$$\{\psi(x), \bar{\psi}(y)\} = \frac{1}{i} S_m(x-y), \quad (140)$$

with

$$S_m(x) = S_m^{(-)}(x) + S_m^{(+)}(x) = (i\not{\partial} + m) D_m(x-y). \quad (141)$$

The anticommutators between two ψ 's and two $\bar{\psi}$'s vanish.

3.1.3 Vector Fields

Next we consider the massless vector field which obeys the wave equation $\square A^\mu(x) = 0$. Its Fourier decomposition reads

$$A^\mu(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega}} \left(a^\mu(\vec{k}) e^{-ikx} + a^\mu(\vec{k})^\dagger e^{ikx} \right), \quad (142)$$

and it is quantized in Lorentz-invariant form according to

$$[A^\mu(x), A^\nu(y)] = g^{\mu\nu} i D_0(x-y). \quad (143)$$

We need also the commutators of the absorption and emission parts alone

$$[A_-^\mu(x), A_+^\nu(y)] = g^{\mu\nu} i D_0^{(+)}(x-y), \quad (144)$$

$$[A_+^\mu(x), A_-^\nu(y)] = g^{\mu\nu} i D_0^{(-)}(x-y). \quad (145)$$

We are working in the Feynman gauge for the sake of convenience and covariance. However, asymptotic massless spin 1 particles only have two polarization degrees of freedom. Consequently, the four polarization types of emission operators introduced above create unphysical particle states, and the space of physical states is a subspace of the full Fock-Hilbert space.

This observation is closely related to the issue of gauge transformations, therefore we also comment here on gauge transformations in perturbative quantum field theory. The massless vector fields describe non-interacting photons and gluons in the standard model. In classical electrodynamics the vector potential can be changed by a gauge transformation

$$A'^\mu(x) = A^\mu(x) + \lambda \partial^\mu u(x), \quad (146)$$

where $u(x)$ is assumed to fulfill the wave equation $\square u(x) = 0$ because we want the transformed field $A'^\mu(x)$ to satisfy the wave equation also. In QFT, the quantized $A'^\mu(x)$ should fulfill the same commutation relations Eq. (143) as $A^\mu(x)$. This is true if the gauge transformation Eq. (146) is of the following form

$$A'^\mu(x) = e^{-i\lambda Q} A^\mu(x) e^{i\lambda Q}, \quad (147)$$

where Q is some operator in the Fock-Hilbert space. Expanding this by means of the Lie series

$$= A^\mu(x) - i\lambda[Q, A^\mu(x)] + O(\lambda^2). \quad (148)$$

and comparing with Eq. (146) we conclude

$$[Q, A^\mu(x)] = i\partial^\mu u(x). \quad (149)$$

The operator Q will be called gauge charge because it is the infinitesimal generator of the gauge transformation defined by Eq. (146). For the following it is important to have Q nilpotent

$$Q^2 = 0. \quad (150)$$

The important consequence of this property is the fact that the factor space $F_{\text{ph}} = \text{Ker } Q / \overline{\text{Ran } Q}$ is isomorphic to the subspace of physical states. Here, Ran is the range and Ker the kernel of the operator Q . The overline denotes the closure; note that $\text{Ran } Q$ is not closed because 0 is in the essential spectrum of Q . We will not discuss this in detail but refer to the literature [6]. Such a nilpotency according to Eq. (150) is characteristic for Fermi operators. Therefore, we assume $u(x)$ to be a fermionic scalar field with mass zero, a so-called ghost field. This field has the following Fourier decomposition

$$u(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2E}} \left(c_2(\vec{p}) e^{-ipx} + c_1(\vec{p})^\dagger e^{ipx} \right). \quad (151)$$

In addition, we introduce a second scalar field

$$\tilde{u}(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2E}} \left(-c_1(\vec{p}) e^{-ipx} + c_2(\vec{p})^\dagger e^{ipx} \right). \quad (152)$$

The absorption and emission operators c_j, c_k^\dagger obey the anticommutation relations

$$\{c_j(\vec{p}), c_k(\vec{q})^\dagger\} = \delta_{jk} \delta^{(3)}(\vec{p} - \vec{q}). \quad (153)$$

Some remarks are in order here. Firstly, also bosonic fields would do the job in the case of an abelian theory like quantum electrodynamics (QED). However, non-abelian gauge theories like quantum chromodynamics (QCD) require fermionic ghosts. In order to avoid any conflict with the spin-statistics theorem, states containing ghosts necessarily do not belong to the physical sector of the Fock-Hilbert space of the theory under consideration. Secondly, when the spin-1 fields become massive, as it is the case in the standard model for the W^\pm - and Z -boson fields, the corresponding ghost fields also become massive. The ghost mass then depends on the chosen formalism (i.e., the gauge fixing, [8]). In the following, we allow the ghost fields to be massive, but $m = 0$ holds true whenever the related vector fields are massless. The formalism used below is chosen so that the ghost mass is equal to the vector boson mass.

Again, the absorption and emission parts (with the adjoint operators) are denoted by (-) and (+). They satisfy the following anticommutation relations

$$\{u^{(-)}(x), \tilde{u}^{(+)}(y)\} = (2\pi)^{-3} \int \frac{d^3p}{2E} e^{-ip(x-y)} = -iD_m^{(+)}(x-y) \quad (154)$$

$$\{u^{(+)}(x), \tilde{u}^{(-)}(y)\} = -(2\pi)^{-3} \int \frac{d^3p}{2E} e^{ip(x-y)} = -iD_m^{(-)}(x-y). \quad (155)$$

All other anticommutators vanish. This implies

$$\{u(x), \tilde{u}(y)\} = -iD_m(x-y) \quad (156)$$

and $\{u(x), u(y)\} = 0$. Then it is not hard to verify that the nilpotent gauge charge Q satisfying Eq. (149) is given by

$$Q = \int d^3x [\partial_\nu A^\nu \partial_0 u - (\partial_0 \partial_\nu A^\nu) u] \stackrel{\text{def}}{=} \int d^3x \partial_\nu A^\nu \overleftrightarrow{\partial}_0 u \quad (157)$$

where the integrals are taken over any plane $x^0 = \text{const}$.

Now we return to the defining property of Q as being the infinitesimal generator of gauge transformations given by Eq. (147) and Eq. (149). We introduce the notation

$$d_Q F = [Q, F],$$

if F contains only Bose fields and an even number of ghost fields, and

$$d_Q F = \{Q, f\} = QF + FQ, \quad (158)$$

if F contain an odd number of ghost fields. Then d_Q has all properties of an anti-derivation, in particular the identity

$$\{AB, C\} = A\{B, C\} - [A, C]B \quad (159)$$

implies the product rule

$$d_Q(F(x)G(y)) = (d_Q F(x))G(y) + (-1)^{n_F} F(x)d_Q G(y), \quad (160)$$

where n_F is the ghost number of F , i.e. the number of u 's minus the number of \tilde{u} 's. The gauge variations d_Q of our free fields now are

$$d_Q A^\mu = i\partial^\mu u, \quad d_Q A_\pm^\mu = i\partial^\mu u_\pm \quad (161)$$

$$d_Q u = 0, \quad d_Q \tilde{u} = \{Q, \tilde{u}\} = -i\partial_\mu A^\mu, \quad d_Q \tilde{u}_\pm = -i\partial_\mu A_\pm^\mu. \quad (162)$$

The latter follows from the anticommutation relation Eq. (156). d_Q changes the ghost number by one, i.e. a Bose field goes over into a Fermi field and vice versa. Then the nilpotency $Q^2 = 0$ implies for a Bose field F_B

$$d_Q^2 F_B = \{Q, [Q, F_B]\} = Q(QF_B - F_B Q) + (QF_B - F_B Q)Q = 0,$$

and for a Fermi field F

$$d_Q^2 F = [Q, \{Q, F\}] = Q(QF_B + F_B Q) - (QF_B - F_B Q)Q = 0,$$

hence

$$d_Q^2 = 0 \quad (1.4.31)$$

is also nilpotent. The gauge variation d_Q has some similarity with the Becchi-Rouet-Stora-Tyutin (BRST) transformation [33, 34] in the functional approach. However, the BRST transformation operates on interacting fields (mainly classical) and the quantum gauge invariance which we are going to define is completely different from BRST invariance.

Now we consider massive vector fields. These fields will be used to represent the W^\pm - and Z -bosons of the electroweak theory, for example. They obey the Klein-Gordon equation

$$(\square + m^2)A^\mu(x) = 0. \quad (163)$$

Since a spin-1 field has three physical degrees of freedom, we need one subsidiary condition to define unphysical states. As this we can choose the Lorentz condition

$$\partial_\mu A^\mu(x) = 0. \quad (164)$$

The commutation relations are similar to the massless case, for example (see Eq. (143))

$$[A^\mu(x), A^\nu(y)] = g^{\mu\nu} i D_m(x - y). \quad (165)$$

where only the massless D -distributions must be substituted by massive ones.

As in the massless case we would like to characterize the physical subspace with help of a nilpotent gauge charge Q . The old definition given by Eq. (157) does not work because nilpotency is violated:

$$Q^2 = \frac{1}{2} \{Q, Q\} = \frac{1}{2} i \int d^3x (\square u) \overleftrightarrow{\partial}_0 u = -\frac{1}{2} i m^2 \int d^3x u \overleftrightarrow{\partial}_0 u \neq 0. \quad (166)$$

To restore it we modify the expression for Q by introducing a scalar field $\Phi(x)$ with the same mass m as the gauge field $A^\nu(x)$

$$Q = \int d^3x (\partial_\nu A^\nu + m\Phi) \overleftrightarrow{\partial}_0 u. \quad (167)$$

All fields satisfy the Klein-Gordon equation

$$(\square + m^2)\Phi = 0 \quad (168)$$

$$(\square + m^2)u = 0, \quad (169)$$

but, while $u(x)$ is a Fermi field, $\Phi(x)$ is quantized with commutation relations

$$[\Phi(x), \Phi(y)] = -i D_m(x - y), \quad (170)$$

and all other commutators are the same as before. Now we can check the nilpotency:

$$Q^2 = -\frac{1}{2} \int d^3x [\partial_\nu A^\nu + m\Phi, Q] = 0, \quad (171)$$

because the first term in the commutator gives $-i\square u$ and the second one $-im^2 u$, so that the sum is zero by Eq. (169). The infinitesimal gauge transformations or gauge variations in the massive case are now given by

$$d_Q A^\mu(x) = [Q, A^\mu(x)] = i\partial^\mu u(x) \quad (172)$$

$$d_Q \Phi(x) = [Q, \Phi(x)] = imu(x) \quad (173)$$

$$d_Q u(x) = \{Q, u(x)\} = 0 \quad (174)$$

$$d_Q \tilde{u}(x) = \{Q, \tilde{u}(x)\} = -i(\partial_\mu A^\mu + m\Phi(x)). \quad (175)$$

The last equation follows from Eq. (167); using $Q^2 = 0$ from Eq. (175) implies Eq. (173).

Let us stress the difference between our approach to massive gauge fields and the conventional one. In the usual approach one starts with massless gauge fields and the scalar field Φ is the so-called Goldstone boson. The fields become massive after "spontaneous breaking" of the gauge symmetry. We start directly with massive vector fields. To define a gauge variation with a nilpotent Q , we are forced to introduce the scalar field Φ , spontaneous symmetry breaking plays no immediate rôle. There is a *common misconception* in the literature, that the Higgs field "gives mass" to the particles. One could also argue that if particles are massive, then a consistent theory requires additional degrees of freedom, i.e. a Higgs sector.

3.1.4 Spin-2 Fields

Finally, we comment on spin-2 quantum gauge theories which can be analyzed on the same footing as spin-1 theories. We only consider the massless case which is relevant for quantum gravity. We start from a symmetric tensor field $h^{\alpha\beta}(x)$ with arbitrary trace which is assumed to satisfy the wave equation

$$\square h^{\alpha\beta}(x) = 0 \quad (176)$$

The gauge transformation similar to Eq. (146) is of the form

$$\tilde{h}^{\alpha\beta} = h^{\alpha\beta} + \lambda(u^{\alpha,\beta} + u^{\beta,\alpha} - g^{\alpha\beta}u^\mu{}_{,\mu}), \quad (177)$$

where the comma denotes partial derivatives. This transformation leaves the so-called Hilbert condition $h^{\alpha\beta}{}_{,\beta} = 0$ unchanged, if u^α fulfills the wave equation

$$\square u^\alpha = 0. \quad (178)$$

The Hilbert gauge condition is analogous to the Lorentz condition in the spin-1 case. The corresponding gauge charge can immediately be written down in analogy to Eq. (157):

$$Q = \int d^3x h^{\alpha\beta}{}_{,\beta} \overleftrightarrow{\partial}_0 u_\alpha. \quad (179)$$

The vector field u_α must be quantized with anticommutators, in order to get Q nilpotent. The operator Q given by Eq. (179) is the right infinitesimal generator for Eq. (177) if it has the following commutator

$$[Q, h^{\alpha\beta}(x)] = -\frac{i}{2} \left(u^{\alpha,\beta} + u^{\beta,\alpha} - g^{\alpha\beta}u^\mu{}_{,\mu} \right)(x) \stackrel{\text{def}}{=} -ib^{\alpha\beta\mu\nu}u_{\mu,\nu} = d_Q h^{\alpha\beta}. \quad (180)$$

The factor $-i/2$ is convention and the b -tensor

$$b^{\alpha\beta\mu\nu} = \frac{1}{2}(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu}) \quad (181)$$

often appears in connection with tensor fields. It is also the four-dimensional extension of DeWitt's supermetric [9]. The commutator Eq. (180) implies the following commutation relation for the tensor field

$$[h^{\alpha\beta}(x), h^{\mu\nu}(y)] = -ib^{\alpha\beta\mu\nu}D_0(x-y) \quad (182)$$

$$= -\frac{i}{2}(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu})D_0(x-y). \quad (183)$$

The vector field u^μ must again be quantized with anticommutators

$$\{u^\mu(x), \tilde{u}^\nu(y)\} = ig^{\mu\nu}D_0(x-y) \quad (184)$$

and the anticommutators between two u 's or two \tilde{u} 's vanish. u^μ, \tilde{u}^ν are called vector-ghost fields, respectively. An explicit representations of the ghost fields is given by

$$\begin{aligned} u^\nu(x) &= (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left(+c_2^\nu(\vec{p})e^{-ipx} - g^{\nu\lambda}c_1^\lambda(\vec{p})^\dagger e^{ipx} \right), \\ \tilde{u}^\nu(x) &= (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left(-c_1^\nu(\vec{p})e^{-ipx} - g^{\nu\lambda}c_2^\lambda(\vec{p})^\dagger e^{ipx} \right); \end{aligned} \quad (185)$$

where the absorption and creation operators satisfy the commutation relations

$$\{c_j^\mu(\vec{p}), c_k^\nu(\vec{k})^\dagger\} = \delta_{jk} \delta^{\mu\nu} \delta^{(3)}(\vec{p} - \vec{k}). \quad (186)$$

The gauge variation of the vector ghost fields now follows from Eq. (179)

$$d_Q u^\alpha = \{Q, u^\alpha\} = 0 \quad (187)$$

$$d_Q \tilde{u}^\alpha = \{Q, \tilde{u}^\alpha\} = ih^{\alpha\mu}{}_{,\mu}. \quad (188)$$

3.2 The Causal Structure of the Perturbative S-Matrix

3.2.1 General Construction of the S-Matrix

Perturbation theory relies strongly on the axiom of causality, as shown by H.Epstein and V.Glaser [10] after previous work of Stückelberg, Bogoliubov and Shirkov [5]. The S-matrix is constructed inductively order by order as a formal power series of operator valued distributions

$$S(g) = \sum_{n=0}^{\infty} S^{(n)}(g) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n T_n(x_1, \dots, x_n) g(x_1) \dots g(x_n) \quad (189)$$

where $g(x)$ is a tempered test function that switches the interaction. The operator-valued distributions T_n act in the Fock space of some collection of free fields. They are called time-ordered or chronological products and should verify the so-called Bogoliubov axioms.

- (1) It is clear from Eq. (189) that T_n can be assumed to be completely symmetrical in all variables x_1, \dots, x_n .
- (2) We must have Poincaré invariance:

$$U_{a,\Lambda} T_n(x_1, \dots, x_n) U_{a,\Lambda}^{-1} = T_n(\Lambda \cdot x_1 + a, \dots, \Lambda \cdot x_n + a) \quad (190)$$

for all proper Lorentz transformations Λ . In particular, translation invariance is essential in the causal approach.

(3) The central axiom is the requirement of causality which can be written compactly as follows. If $X = \{x_1, \dots, x_m\} \in \mathbb{R}^{4m}$ and $Y = \{y_1, \dots, y_n\} \in \mathbb{R}^{4n}$ are such that $x_i - y_j \notin \overline{V^-}$ for all i and j , we say X is later than Y , $X \geq Y$. We use the compact notation $T_n(X) = T_n(x_1, \dots, x_n)$ and by $X \cup Y$ we mean the union of the elements of X and Y . In particular, the expression $T_{n+m}(X \cup Y)$ makes sense because of the symmetry property (1). Now the causality axiom expresses causal factorization:

$$T_{n+m}(X \cup Y) = T_m(X) T_n(Y), \quad \forall X \geq Y. \quad (191)$$

Physically this means that later action does not influence what has happened before.

Like $S(g)$ given by Eq. (189), the inverse $S(g)^{-1}$ can be expressed by a perturbation series

$$S(g)^{-1} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \tilde{T}_n(x_1 \dots x_n) g(x_1) \dots g(x_n) \quad (192)$$

The corresponding n -point distributions \tilde{T}_n , called anti-chronological products follow from Eq. (189) as formal inversion of a power series

$$\tilde{T}_n(X) = \sum_{r=1}^n (-)^r \sum_{P_r} T_{n_1}(X_1) \dots T_{n_r}(X_r), \quad (193)$$

where the second sum runs over all partitions P_r of X into r disjoint subsets

$$X = X_1 \cup \dots \cup X_r, \quad X_j \neq \emptyset, \quad |X_j| = n_j. \quad (194)$$

All products of distributions in Eq. (193) are well-defined, because the arguments are disjoint sets of points such that the products are direct products of distributions.

- (4) Unitarity of the S-matrix

$$S(g)^{-1} = S(g)^\dagger. \quad (195)$$

can now be expressed by means of the time-ordered products in the form

$$\tilde{T}_n(X) = T_n(X)^\dagger. \quad (196)$$

It is one aim of QFT to prove unitarity for the physically interesting theories. In the inductive construction unitarity is not used.

Now we are ready to turn to the inductive construction of $T_n(x_1 \dots x_n)$ starting from $T_1(x)$ which is a given interaction Lagrangian or coupling. Suppose all $T_m(x_1, \dots, x_m)$ for $1 \leq m \leq n-1$ are known and have the above properties (1)-(3). Then, according to Eq. (193), the $\tilde{T}_m(X)$ can be calculated for all $1 \leq m = |X| \leq n-1$. From this it is possible to form the following distributions

$$A'_n(x_1 \dots x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) \quad (197)$$

$$R'_n(x_1 \dots x_n) = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X), \quad (198)$$

where the sums run over all partitions

$$P_2 : \{x_1, \dots, x_{n-1}\} = X \cup Y, \quad X \neq \emptyset \quad (199)$$

into disjoint subsets with $|X| = n_1 \geq 1$, $|Y| \leq n-2$. We also introduce

$$D_n(x_1, \dots, x_n) = R'_n(x_1, \dots, x_n) - A'_n(x_1, \dots, x_n). \quad (200)$$

If the sums are extended over all partitions P_2^0 , including the empty set $X = \emptyset$, then we get the distributions

$$A_n(x_1, \dots, x_n) = \sum_{P_2^0} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) \quad (201)$$

$$= A'_n + T_n(x_1 \dots x_n), \quad (202)$$

$$R_n(x_1, \dots, x_n) = \sum_{P_2^0} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) \quad (203)$$

$$= R'_n + T_n(x_1 \dots x_n). \quad (204)$$

These two distributions A_n, R_n are not known by the induction assumption because they contain the unknown $T_n(x_1, \dots, x_n)$. Only the difference

$$D_n = R'_n - A'_n = R_n - A_n \quad (205)$$

is known according to Eq. (200). What remains to be done is to determine R_n (or A_n) in Eq. (205) separately. This is achieved by investigating the support properties of the various distributions.

We recall the definition Eq. (65) of the n -dimensional generalizations of the forward and backward light-cones with respect to the point x . Then we have

$$\text{supp } R_{n_1+1}(Y, x) \subseteq \Gamma_{n_1+1}^+(x) \quad (206)$$

and

$$\text{supp } A_{n_1+1}(Y, x) \subseteq \Gamma_{n_1+1}^-(x). \quad (207)$$

Because of these support properties, R and A are called retarded and advanced distributions, respectively. The distribution D , which can be expressed by Eq. (205), then has a causal support:

$$\text{supp } D_n(x_1, \dots, x_{n-1}, x_n) \subseteq \Gamma_n^+(x_n) \cup \Gamma_n^-(x_n). \quad (208)$$

We do not present the proof here (see [7], sect. 3.1) but we can indicate the essential reason for this important causal support property: According to Eq. (197) and Eq. (198) D_n is a sum of commutators

$$D'_n(x_1 \dots x_n) = \sum_{P_2} [T_{n-n_1}(Y, x_n), \tilde{T}_{n_1}(X)]. \quad (209)$$

Since all T 's are products of free fields, the commutators contain Jordan-Pauli distributions which have causal support according to Eq. (56).

Now we see the inductive construction clearly before us: From the known $T_m(x_1, \dots, x_m)$, $m \leq n-1$ one computes A_n given by Eq. (197) and R_n from Eq. (198), and then $D_n = R'_n - A'_n$. One decomposes D_n with respect to the supports Eq. (208)

$$D_n(x_1, \dots, x_n) = R_n(x_1, \dots, x_n) - A_n(x_1, \dots, x_n), \quad (210)$$

$$\text{supp } R_n \subseteq \Gamma_{n-1}^+(x_n) \quad , \quad \text{supp } A_n \subseteq \Gamma_{n-1}^-(x_n). \quad (211)$$

Finally, T_n is found from Eq. (202) or Eq. (204)

$$T_n(x_1, \dots, x_n) = R_n(x_1, \dots, x_n) - R'_n(x_1, \dots, x_n) \quad (212)$$

$$= A_n(x_1, \dots, x_n) - A'_n(x_1, \dots, x_n). \quad (213)$$

The only non-trivial step in this construction is the distribution splitting Eq. (210). In sect. 2.3 we have discussed the splitting of causal numerical distributions. The transformation of the operator-valued distribution D_n to numerical distributions is achieved by means of Wick expansion. D_n can be written in a unique way in terms of normally ordered products of free field operators Ψ

$$D_n = \sum d_{k_1 \dots k_n}(x_1, \dots, x_n) : \Psi_{k_1}(x_1) \dots \Psi_{k_n}(x_n) :, \quad (214)$$

where k_1, \dots, k_n are indices specifying the individual field operator types and the corresponding field components related to external symmetries (e.g., Lorentz indices) and inner symmetries (e.g., color indices), depending on the theory under study.

In the normal product between double dots all absorption operators stand to the right of all emission operators. Consequently, the vacuum expectation value of a normal product vanishes. This allows us to write the normal ordering of, e.g., n scalar fields φ in the compact form

$$\varphi(x_1) \dots \varphi(x_n) = \sum_{\substack{s_1, \dots, s_n \\ s_j=0,1}} (\Omega, \varphi^{1-s_1}(x_1) \dots \varphi^{1-s_n}(x_n) \Omega) : \varphi^{s_1}(x_1) \dots \varphi^{s_n}(x_n) :, \quad (215)$$

and the generalization to products of general free field operators is straightforward. Here Ω is the vacuum in Fock space and the brackets (\cdot, \cdot) mean the scalar product. Note that all terms with an odd number of $s_j = 0$ are zero because the vacuum expectation value vanishes.

The splitting of D_n in Eq. (214) can now be carried out by splitting all numerical distributions d_k . We have learned in sect. 2.3 that the splitting solutions may be not unique. Then the free but finite local terms must be chosen such that all properties required for the S-matrix are true. This is a subtle problem for gauge theories.

3.2.2 Example

As a simple illustration of the causal method we consider the coupling

$$T_1(x) = -i\lambda : \varphi^\dagger(x) \varphi(x) : \Phi(x) \quad (216)$$

between a charged scalar field φ of mass m and a neutral scalar Φ of mass M . This theory has the same structure as quantum electrodynamics: φ is a spin-0 electron and Φ a scalar massive photon. To perform the normal ordering according to Eq. (215) we need the commutators, or contractions

$$\overline{\varphi(x) \varphi^\dagger(y)} = [\varphi(x)^{(-)}, \varphi^\dagger(y)^{(+)})] = (\Omega, \varphi(x) \varphi^\dagger(y)) = -iD_m^{(+)}(x-y) \quad (217)$$

and similarly for Φ .

In the inductive step from T_1 to T_2 we must first compute

$$R'_2(x_1, x_2) = -T_1(x_2)T_1(x_1) \quad (218)$$

by normal ordering:

$$\begin{aligned} &= \lambda^2 \left(: \varphi^\dagger(x_1)\varphi(x_1)\varphi^\dagger(x_2)\varphi(x_2) :: \Phi(x_1)\Phi(x_2) : \right. \\ &\quad - iD_m^{(+)}(x_2 - x_1) : \varphi^\dagger(x_1)\varphi(x_2) :: \Phi(x_1)\Phi(x_2) : \\ &\quad - iD_m^{(+)}(x_2 - x_1) : \varphi(x_1)\varphi^\dagger(x_2) :: \Phi(x_1)\Phi(x_2) : \\ &\quad - iD_M^{(+)}(x_2 - x_1) : \varphi^\dagger(x_1)\varphi(x_1)\varphi^\dagger(x_2)\varphi(x_2) : \\ &\quad - D_m^{(+)}(x_2 - x_1)D_m^{(+)}(x_2 - x_1) : \Phi(x_1)\Phi(x_2) : \\ &\quad - D_m^{(+)}(x_2 - x_1)D_M^{(+)}(x_2 - x_1) : \varphi^\dagger(x_1)\varphi(x_2) : \\ &\quad - D_m^{(+)}(x_2 - x_1)D_M^{(+)}(x_2 - x_1) : \varphi(x_1)\varphi^\dagger(x_2) : \\ &\quad \left. - iD_m^{(+)}(x_2 - x_1)D_m^{(+)}(x_2 - x_1)D_M^{(+)}(x_2 - x_1) \right). \end{aligned} \quad (219)$$

We emphasize that here the product of two $D^{(+)}(x)D^{(+)}(x)$ is well defined, in contrast to the product of two Feynman propagators $D_F(x)$ in sect. 2.1. The reason is that in the Fourier-transformed expression of, e.g.,

$$\frac{1}{(2\pi)^2} \int d^4q \hat{D}_m^{(+)}(p - q)\hat{D}_m^{(+)}(q) \quad (220)$$

the intersection of the supports of the two $\hat{D}_m^{(+)}$ is a compact set. This can be easily understood if one remembers that the support in momentum space of the two individual Pauli-Jordan distributions in Eq. (220) is contained in a forward and a backward light-cone, respectively.

In the same way A'_2 can be computed and then $D_2 = R'_2 - A'_2$. From the third line in Eq. (219) we get the following contribution to D_2 :

$$D_2^1 = -i\lambda^2 D_m(x_2 - x_1) : \varphi(x_1)\varphi^\dagger(x_2) :: \Phi(x_1)\Phi(x_2) : . \quad (221)$$

Here D_m can be trivially split. The retarded part with respect to x_2 contains $D_m^{av}(x_2 - x_1)$. Adding $R_2 \sim D_m^{(+)}$ and using $D_m^{av} + D_m^{(+)} = D_F^m$ we finally obtain

$$T_2^1 = i\lambda^2 D_F^m(x_2 - x_1) : \varphi(x_1)\varphi^\dagger(x_2) :: \Phi(x_1)\Phi(x_2) : . \quad (222)$$

This gives "electron-photon" scattering in this model. The one contraction between the vertices x_1 and x_2 is represented by the Feynman propagator. So in tree graphs the usual Feynman rules hold.

Now let us consider a loop graph with two contractions, for example "vacuum polarization" which comes from the fifth term in Eq. (219). The corresponding causal distribution is given by

$$D_2^2(x_1, x_2) = \lambda^2 [D_m^{(+)}(y)D_m^{(+)}(y) - D_m^{(+)}(-y)D_m^{(+)}(-y)] : \Phi(x_1)\Phi(x_2) : , \quad (223)$$

where $y = x_1 - x_2$. We calculate

$$d_+(-y) \stackrel{\text{def}}{=} D_m^{(+)}(y)D_m^{(+)}(y) \quad (224)$$

in momentum space:

$$\hat{d}_+(k) = -(2\pi)^{-4} \int d^4p \Theta(k_0 - p_0)\delta((k - p)^2 - m^2)\Theta(p_0)\delta(p^2 - m^2). \quad (225)$$

It is easy to evaluate this for time-like k in the special Lorentz frame such that $k = (k_0, \vec{0})$. The result for arbitrary k is then

$$\hat{d}_+(k) = -\frac{(2\pi)^{-3}}{4} \Theta(k_0) \Theta(k^2 - 4m^2) \sqrt{1 - \frac{4m^2}{k^2}}. \quad (226)$$

The total result for the Fourier transform of the square bracket in Eq. (223) denoted by $\hat{d}(k)$ is simply obtained by substituting $\Theta(k_0)$ by $\text{sgn}(k_0)$.

Since $\hat{d}(k)$ has a constant quasi-asymptotics it has singular order $\omega = 0$. Consequently, the distribution splitting is non-trivial. Therefore, the two internal lines in the vacuum-energy graph cannot be represented by Feynman propagators; the Feynman rules are not true for loop graphs. Instead we must use the dispersion integral Eq. (104) with $\omega = 0$:

$$\hat{r}(k_0) = \frac{i}{2\pi} k_0 \int_{-\infty}^{+\infty} \frac{dp_0}{(p_0 - i0)(k_0 - p_0 + i0)} \quad (227)$$

$$\cdot \left[-\frac{(2\pi)^{-3}}{4} \Theta(p^2 - 4m^2) \sqrt{1 - \frac{4m^2}{p^2}} \text{sgn}(p_0) \right]. \quad (228)$$

For the time-ordered product we have to calculate $\hat{t}(k_0) = \hat{r}(k_0) - \hat{r}'(k_0)$. $r'(k_0) \sim \hat{d}_+(-k_0)$ nicely combines with Eq. (228) so that

$$\hat{t}(k_0) = -\frac{i}{(2\pi)^4} \frac{k_0^2}{4} \int_{4m^2}^{+\infty} ds \frac{1}{s(k_0 - s + i0)} \sqrt{1 - \frac{4m^2}{s}}. \quad (229)$$

This integral is elementary, the final result for time-like momentum $k_0 = \sqrt{-k^2}$ is equal to

$$\hat{t}(k_0) = \frac{i}{(2\pi)^4} \left[1 + \frac{1 + \left(\frac{ik_0}{2m} - \sqrt{1 - \frac{k_0^2}{4m^2}} \right)^2}{1 - \left(\frac{ik_0}{2m} - \sqrt{1 - \frac{k_0^2}{4m^2}} \right)^2} \log \left(\frac{ik_0}{2m} - \sqrt{1 - \frac{k_0^2}{4m^2}} \right) \right]. \quad (230)$$

The result for space-like k is obtained by analytic continuation. We leave it as an exercise to the reader to compare this result to the alternative outcome given below at the end of sect. 4.2.

4 Regularization Methods

4.1 Basic remarks

It would not be worthwhile to recapitulate the well-known details of the different regularization methods which are on the market. While the Pauli-Villars regularization can be considered as an ad hoc method to solve the apparent problem of infinities in perturbative quantum field theory, dimensional regularization is treated by many introductory texts like [11, 12], including the original works on the topic [13, 14].

Dimensional regularization regularizes Feynman diagrams by analytic continuation to $4 - \epsilon$ (complex) space-time dimensions and isolates infrared and ultraviolet divergences as poles in ϵ . From a technical point of view, the main question is how to evaluate Feynman diagrams in n dimensions, i.e. it is necessary to know the usual Feynman rules of the theory, properties of Dirac matrices in n dimensions, and techniques like the Feynman parametrization for performing the momentum integrals in n dimensions. Difficulties may arise when one has to deal with topological quantities which exist only in integer dimensions.

The same applies to the causal method, however, there is a big conceptual difference between the causal and dimensional regularization approach to perturbative quantum field theory. Whereas the causal approach is a

mathematically fully understood perturbative method, is hard to give a precise meaning to the idea of physics in an arbitrary complex number of space-time dimensions. Still, the method has many interesting technical advantages, and although a proof of the physical equivalence of the causal and dimensional regularization method is lacking due to the technical complexity of the problem, one should not be too pessimistic about that issue.

In the forthcoming section, we will illustrate to conceptual differences by presenting some specific examples in the light of the different approaches. There, the diagrams will turn out to be finite in most cases, but this is not the central issue since infinities can always be removed in one or the other way. In the present short section, we present a direct comparison of the treatment of the scalar one-loop integral in the causal and dimensional regularization approach.

4.2 Scalar One-Loop Diagram in n Dimensions

The positive frequency part of the Pauli-Jordan distribution in n dimensions is given in the causal framework by

$$D_m^{(+)}(x) = \frac{i}{(2\pi)^{n-1}} \int d^n p \delta(p^2 - m^2) \Theta(p_0) e^{-ipx}. \quad (231)$$

Strictly speaking, the expression above is well-defined in integer dimensions. In order to obtain an analytic expression for the scalar one-loop diagram in arbitrary dimensions, we generalize Eq. (225) to

$$\begin{aligned} \hat{d}_+^{(n)}(k) &= \frac{i^2}{(2\pi)^{n/2+2n-2}} \int d^n k e^{ikx} \int d^n p_1 e^{-ip_1 x} \int d^n p_2 e^{-ip_2 x} \Theta(p_1^0) \delta(p_1^2 - m^2) \Theta(p_2^0) \delta(p_2^2 - m^2) \\ &= -\frac{1}{(2\pi)^{3n/2-2}} \int d^n p \Theta(k^0 - p^0) \delta((k-p)^2 - m^2) \Theta(p^0) \delta(p^2 - m^2). \end{aligned} \quad (232)$$

We exploit the last Θ - and δ -distributions in Eq. (232) and evaluate the integral over space-like momenta ($E = \sqrt{\vec{p}^2 + m^2}$, $k^0 = k_0$)

$$I^{(n)} = \int \frac{d^{n-1} p}{2E} \delta(k_0^2 - 2k_0 E) \Theta(k^0 - E) \quad (233)$$

and obtain a radial integral, using $E = k^0/2$, $|\vec{p}| = \sqrt{k_0^2/4 - m^2}$

$$I^{(n)} = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \Theta(k_0^2 - 4m^2) \int d|\vec{p}| \frac{|\vec{p}|^{n-2}}{2E} \delta(2k_0(k^0/2 - E)) \Theta(k^0 - E), \quad (234)$$

since the $(n-2)$ -dimensional surface of an $(n-1)$ -dimensional unit ball is given by $(2\pi^{(n-1)/2})/\Gamma((n-1)/2)$. Substituting $|\vec{p}|d|\vec{p}| = E dE$, $I^{(n)}$ can be written

$$\begin{aligned} I^{(n)} &= \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \Theta(k_0^2 - 4m^2) \int dE |\vec{p}|^{n-3} \frac{\delta(E - k^0/2)}{2k^0} \Theta(k^0/2) \\ &= \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \Theta(k_0^2 - 4m^2) \Theta(k^0) \frac{\sqrt{k_0^2/4 - m^2}^{n-3}}{2k^0} \end{aligned} \quad (235)$$

or

$$\hat{d}_+^{(n)}(k) = -\frac{\pi^{(n-1)/2}}{(2\pi)^{3n/2-2} \Gamma((n-1)/2)} \Theta(k^2 - 4m^2) \Theta(k^0) \frac{\sqrt{k^2/4 - m^2}^{n-3}}{2\sqrt{k^2}}. \quad (236)$$

This result can be compared directly to Eq. (225) for $n = 4$.

Now, the interesting point is that the $(1 - t + i0)$ -term in the central splitting formula Eq. (105) generates the real part of the scalar loop amplitude, denoted here by $\hat{t}^{(n)}(k)$, since

$$\frac{1}{1 - t + i0} = P \frac{1}{1 - t} - i\pi\delta(1 - t), \quad (237)$$

where the symbol P denotes the Cauchy principal value in the sense of distributions. We obtain for k in the forward light-cone

$$\Re(\hat{t}^{(n)}(k)) \sim \Theta(k^2 - 4m^2) \frac{\sqrt{k^2/4 - m^2}^{n-3}}{2\sqrt{k^2}}, \quad (238)$$

where we have omitted numerical factors. In fact, this result is valid for arbitrary momenta k .

Now the scalar loop integral in n dimensions is given by the expression

$$\mathcal{I}^{(n)}(p) = \int \frac{d^n k}{(k^2 - m^2 + i0) [(k - p)^2 - m^2 + i0]} \quad (239)$$

Feynman parametrization

$$\frac{1}{AB} = \int_0^1 d\alpha [\alpha A + (1 - \alpha)B]^{-2} \quad (240)$$

and a subsequent momentum translation $k^\mu \mapsto k^\mu + \alpha p^\mu$ leads to the formal integral

$$\mathcal{I}^{(n)}(p) = \int d^n k \int_0^1 d\alpha [k^2 - m^2 + \alpha(1 - \alpha)p^2 + i0]^{-2}. \quad (241)$$

To evaluate this integral, we may use the relation

$$\int \frac{d^n k}{(k^2 - a^2 + i0)^m} = i \pi^{\frac{n}{2}} \frac{\Gamma(2 - \frac{n}{2})}{(a^2 - i0)^{2 - \frac{n}{2}}}, \quad (242)$$

which is divergent for $n \geq 4$. In the present case, we have

$$a^2 = m^2 - \alpha(1 - \alpha)p^2. \quad (243)$$

The calculation of the finite integral in the case $n = 3$ is discussed in detail in sect. 5.3. We focus here on the case $n = 4$. Note that the following manipulations are formal to some extent. Since

$$\Gamma\left(2 - \frac{n}{2}\right) = \frac{2}{4 - n} - \gamma - \left(\frac{\pi^2}{24} + \frac{\gamma^2}{4}\right)(n - 4) + \dots \quad (244)$$

and

$$a^{n-4} = 1 + (n - 4) \log a + \dots, \quad (245)$$

where γ is the Euler-Mascheroni constant, we obtain in the limit $n \rightarrow 4$

$$\mathcal{I}^{(4)}(p) \rightarrow \frac{2i\pi^2}{4 - n} - i\pi^2 \int_0^1 d\alpha \log(m^2 - \alpha(1 - \alpha)p^2 - i0) - i\pi^2\gamma - i\pi^2 \log(\pi). \quad (246)$$

A relevant finite part of the integral above is given by

$$\mathcal{I}_{reg}^{(4)}(p) = -i\pi^2 \int_0^1 d\alpha \log\left(\frac{m^2 - \alpha(1 - \alpha)p^2 - i0}{m^2}\right)$$

$$= -i\pi^2 \int_0^1 d\alpha \left(\frac{\alpha(2\alpha - 1)}{m^2 - \alpha(1 - \alpha)p^2 - i0} \right), \quad (247)$$

where an integration by parts was performed. For $p^2 > 4m^2$, the $-i0$ -term generates a real part of $\mathcal{I}_{reg}^{(4)}(p)$, which is obtained in a trivial manner from the partial fraction decomposition

$$\frac{1}{\alpha - \alpha_1 - i0} + \frac{1}{\alpha - \alpha_2 + i0} = \frac{2\alpha - 1}{\alpha^2 - \alpha + m^2/p^2 - i0} \quad (248)$$

with

$$\alpha_{1,2} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4m^2}{p^2}}, \quad 0 < \alpha_{1,2} < 1, \quad (249)$$

leading to

$$\Re(\mathcal{I}_{reg}^{(n)}(p)) \sim (\alpha_1 - \alpha_2)\Theta(p^2 - 4m^2) = \Theta(p^2 - 4m^2) \sqrt{1 - \frac{4m^2}{p^2}}, \quad (250)$$

in accordance with Eq. (238). For $p^2 > 4m^2$, one can also write

$$\mathcal{I}_{reg}^{(4)}(p) = i\pi^2 \sqrt{1 - \frac{4m^2}{p^2}} \left(\log \left(\frac{1 - \sqrt{1 - 4m^2/p^2}}{1 + \sqrt{1 - 4m^2/p^2}} \right) + i\pi \right) + 2i\pi^2. \quad (251)$$

This solution is normalized according to

$$\lim_{p \rightarrow 0} \mathcal{I}_{reg,an}^{(4)}(p) = 0, \quad (252)$$

i.e. it corresponds to the central splitting solution in the causal approach when continued analytically to arbitrary p .

We observe that the real part of the scalar loop diagram coincides both for the causal and the dimensional approach, and it is straightforward to show that this result holds in arbitrary dimensions. Furthermore, Eq. (236) provides a kind of "dimensional" generalization of the causal method. From the real part, the imaginary part of the amplitude is obtained from the dispersive splitting formula in the causal approach or by direct computation according to the rules of dimensional regularization. Up to finite renormalizations, the finite parts of the results also agree.

5 Comparison of Regularization Methods to the Causal Approach: Specific Examples

5.1 Axial Anomalies

Axial or triangle anomalies are a subtle problem because their treatment by regularization of divergent Feynman integrals is unsatisfactory. One gets the impression that the anomalies are a consequence of the ultraviolet regularization. Then the question remains whether by some other method of calculating the divergent integral the anomaly might disappear. The causal method is free of such uncertainties as we are going to show.

We consider QED with pseudovector and pseudoscalar couplings

$$\begin{aligned} T_1(x) &= i c_V j_V^\mu(x) A_\mu(x) + i c_A j_A^\mu(x) B_\mu(x) \\ &\quad + i c_\pi j_\pi(x) \Pi(x). \end{aligned} \quad (253)$$

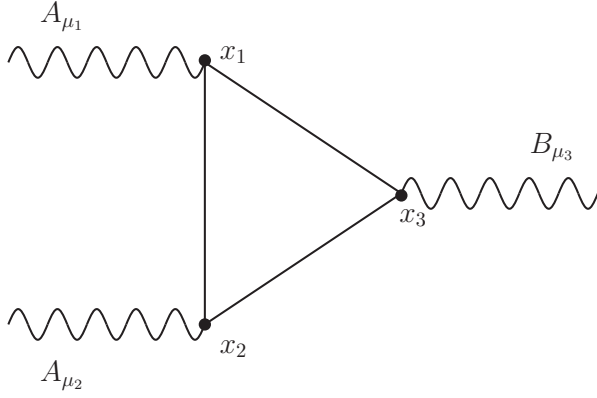


Figure 1: Axial vector triangular graph

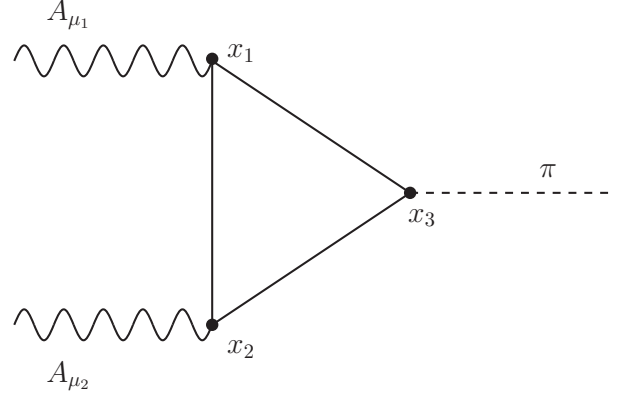


Figure 2: Pseudoscalar triangular graph

Here

$$j_V^\mu =: \bar{\psi} \gamma^\mu \psi : \quad , \quad j_A^\mu =: \bar{\psi} \gamma^\mu \gamma^5 \psi : \quad (254)$$

are the vector and axial vector currents and

$$j_\pi = i : \bar{\psi} \gamma^5 \psi : \quad (255)$$

is a pseudoscalar, all being formed from a free massive Dirac field $\psi(x)$ with mass m . The vector-, axial vector and pseudoscalar vertices defined by Eq. (253) will be abbreviated by V , A and Π in the following. The fields A_μ , B_μ and $\Pi(x)$ play no essential rôle and are, therefore, assumed as classical external fields.

From Eq. (254) and Eq. (255) we have the following divergence relations for the free currents

$$\partial_\mu j_V^\mu = 0, \quad \partial_\mu j_A^\mu = 2m j_\pi, \quad (256)$$

as a consequence of the Dirac equation. Our problem is whether similar divergence relations hold at higher orders, in particular for the two triangular graphs with vertices VVA - and $VV\Pi$ of Figs. (1) and (2) which contribute to the 3-point function T_3 . To compute the latter, we must first calculate

$$\begin{aligned} D_3(x_1, x_2, x_3) &= T_2(x_1, x_3) T_1^\dagger(x_2) + T_2(x_2, x_3) T_1^\dagger(x_1) + T_1(x_3) T_2^\dagger(x_1, x_2) \\ &\quad - T_1^\dagger(x_1) T_2(x_2, x_3) - T_1^\dagger(x_2) T_2(x_1, x_3) - T_2^\dagger(x_1, x_2) T_1(x_3), \end{aligned} \quad (257)$$

where we have used unitarity to express the T -distributions of the inverse S-matrix. Concerning the triangle graphs, the 2-point distributions which contribute, come from Compton scattering. For these distributions the usual divergence relations in Eqns. (256) still hold, so that

$$\frac{\partial}{\partial x_3^{\mu_3}} d_B^{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3) = 2m \frac{c_A}{c_\pi} d_\pi^{\mu_1 \mu_2}(x_1, x_2, x_3). \quad (258)$$

Here d_B , d_π are the numerical 3-point distributions corresponding to the triangle graphs displayed in Figs. (1) and (2) without the external fields A , B , Π . The question is whether the same relation remains true after splitting for the retarded distributions. One therefore defines the anomaly by

$$a^{\mu_1 \mu_2} = \frac{\partial}{\partial x_3^{\mu_3}} r_B^{\mu_1 \mu_2 \mu_3}(x_1, x_2, x_3) - 2m \frac{c_A}{c_\pi} r_\pi^{\mu_1 \mu_2}(x_1, x_2, x_3). \quad (259)$$

The t -distributions have the same anomaly because the r' -distributions are anomaly-free.

Since we work with massive Fermi fields, we can perform the splitting in momentum space by means of the central solution

$$\hat{r}_B(p, q) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{d}_B(tp, tq)}{(1-t+i0)t^{\omega_B+1}} dt \quad (260)$$

where p, q are assumed to be in the forward cone V^+ , and similarly for \hat{r}_π . ω is the singular order of the d -distributions. The Fourier transformation is carried out in the difference variables $y_1 = x_1 - x_3, y_2 = x_2 - x_3$, taking translation invariance into account. From Eq. (258) we then get

$$i(p_{\mu_3} + q_{\mu_3})\hat{d}_B^{\mu_1\mu_2\mu_3}(p, q) = 2m \frac{c_A}{c_\pi} \hat{d}_\pi^{\mu_1\mu_2}(p, q), \quad (261)$$

and the anomaly in Eq. (259) becomes

$$\hat{a}^{\mu_1\mu_2}(p, q) = i(p_{\mu_3} + q_{\mu_3})\hat{r}_B^{\mu_1\mu_2\mu_3} - 2m \frac{c_A}{c_\pi} \hat{r}_\pi^{\mu_1\mu_2}. \quad (262)$$

Substituting Eq. (260) and the analogous equation for \hat{r}_π herein, and using Eq. (261), we arrive at the following formula for the anomaly

$$\hat{a}^{\mu_1\mu_2}(p, q) = \frac{i c_A}{\pi c_\pi} m \int_{-\infty}^{\infty} dt \frac{\hat{d}_\pi^{\mu_1\mu_2}(tp, tq)}{1-t+i0} \left(\frac{1}{t^{\omega_B+2}} - \frac{1}{t^{\omega_\pi+1}} \right). \quad (263)$$

Hence, the anomaly is due to the fact that $\omega_\pi - \omega_B \neq 1$.

To evaluate Eq. (263) we only need the pseudoscalar d -distribution. From the first three terms in Eq. (257) we find

$$\begin{aligned} r'^{\mu_1\mu_2}(y_1, y_2) &= c_V^2 c_\pi \text{tr} [i\gamma_5 S_m^{(+)}(x_3 - x_2) \gamma^{\mu_2} S_{AF}^m(x_2 - x_1) \gamma^{\mu_1} S_m^{(-)}(x_1 - x_3) \\ &\quad + i\gamma_5 S_m^{(+)}(x_3 - x_2) \gamma^{\mu_2} S_m^{(-)}(x_2 - x_1) \gamma^{\mu_1} S_F(x_1 - x_3)^m \\ &\quad + i\gamma_5 S_F^m(x_3 - x_2) \gamma^{\mu_2} S_m^{(+)}(x_2 - x_1) \gamma^{\mu_1} S_m^{(-)}(x_1 - x_3)] + \text{tr} [x_1 \leftrightarrow x_2, \mu_1 \leftrightarrow \mu_2]. \end{aligned} \quad (264)$$

Here S_{AF}^m denotes the anti-Feynman propagator which is obtained from the Feynman propagator in momentum space by changing $+i0$ into $-i0$:

$$\hat{S}_{AF}^m(p) = (2\pi)^{-2} \frac{\not{p} + m}{p^2 - m^2 - i0}. \quad (265)$$

It comes from the adjoint in Eq. (257). If one replaces $S_m^{(+)}$ by $S_m^{(-)}$ and vice versa without changing the arguments, one gets $a'^{\mu_1\mu_2}$. The difference $r' - a'$ gives $d^{\mu_1\mu_2}$.

Expressing the spinor distributions by scalar ones, we see that the terms with three $\not{\partial}$ contain γ_5 plus five other γ -matrices. Then the trace vanishes. In the non-vanishing terms one has at least one factor m instead of $\not{\partial}$. This lowers ω by one, so that we get $\omega_\pi = 0$, instead of the power-counting estimate 1. But we will see below that the splitting with $\omega = 0$ or 1 gives the same result. If we replace $i\gamma_5$ by $\gamma^{\mu_3}\gamma_5$ in Eq. (265), we get the r' -distribution for the axial vector graph: $r'^{\mu_1\mu_2\mu_3}$. Then the terms with three $\not{\partial}$ contain γ_5 plus six γ^μ matrices and the trace does not vanish. In this case we have the power-counting result $\omega_B = 1$.

After Fourier transformation

$$\hat{r}'^{\mu_1\mu_2}(p, q) = (2\pi)^{-4} \int r'^{\mu_1\mu_2}(y_1, y_2) e^{i(py_1 + qy_2)} dy_1 dy_2 \quad (266)$$

we shall obtain

$$\hat{r}'^{\mu_1\mu_2}(p, q) = \frac{c_V^2 c_\pi}{(2\pi)^2} \int dk \left\{ \text{tr} [-i\gamma_5 \hat{S}_m^{(+)}(-P+k) \gamma^{\mu_2} \hat{S}_{AF}^m(-p+k) \gamma^{\mu_1} \hat{S}_m^{(-)}(k) \right.$$

$$\begin{aligned}
& +i\gamma_5 \hat{S}_m^{(+)}(-P+k)\gamma^{\mu_2} \hat{S}_m^{(-)}(-p+k)\gamma^{\mu_1} \hat{S}_F(k)^m \\
& +i\gamma_5 \hat{S}_F^m(-P+k)\gamma^{\mu_2} \hat{S}_m^{(+)}(-p+k)\gamma^{\mu_1} \hat{S}_m^{(-)}(k) + \text{tr} [p \longleftrightarrow q, \mu_1 \longleftrightarrow \mu_2] \Big\}. \tag{267}
\end{aligned}$$

Here we have introduced $P = p + q$. Computing the trace we get

$$\begin{aligned}
\hat{r}'^{\mu_1\mu_2}(p, q) &= -\frac{4m c_V^2 c_\pi}{(2\pi)^6} \varepsilon^{\mu_1\mu_2\alpha\beta} p_\alpha q_\beta \times \\
&\times \{ [I_-(P, p) + I_+(q, -p) + I_+(p, P)] + [p \longleftrightarrow q] \}, \tag{268}
\end{aligned}$$

where the Lorentz invariant integrals I_\pm are given by

$$\begin{aligned}
I_\pm(p, q) &\stackrel{\text{def}}{=} \int d^4k \Theta(-k^0) \delta(k^2 - m^2) \times \\
&\times \Theta(k^0 - p^0) \delta[(k-p)^2 - m^2] \frac{1}{(k-q)^2 - m^2 \pm i0}. \tag{269}
\end{aligned}$$

Owing to the two δ - and Θ -functions, these integrals vanish if p is not in the region $p^2 \geq 4m^2, p_0 < 0$. But if p is in this region, one can use \mathcal{L}_+^\uparrow invariance

$$I_\pm(\Lambda p, \Lambda q) = I_\pm(p, q), \quad \forall \Lambda \in \mathcal{L}_+^\uparrow, \tag{270}$$

to choose $\Lambda p = (-\sqrt{p^2}, \vec{0})$. Then the integration is done as follows: first we integrate over k^0 , using the first δ . In the spatial integration $d^3\vec{k}$, we use polar coordinates with \vec{q} as polar axis. Integration over $|\vec{k}|$ kills the second δ , while the integral over the azimuth φ gives trivially 2π . The remaining integration over $\cos\vartheta = \vec{k} \cdot \vec{q} / (|\vec{k}||\vec{q}|)$ is elementary. The result in an arbitrary Lorentz system is equal to

$$\begin{aligned}
I_\pm(p, q) &= \frac{\pi}{4} \Theta(-p^0) \Theta(p^2 - 4m^2) \frac{1}{\sqrt{N}} \times \\
&\times \log\left(\frac{-pq + q^2 + \sqrt{(1 - 4m^2/p^2)N} \pm i0}{-pq + q^2 - \sqrt{(1 - 4m^2/p^2)N} \pm i0}\right), \tag{271}
\end{aligned}$$

where

$$N = N(p, q) = (pq)^2 - p^2 q^2. \tag{272}$$

The final result for Eq. (268) is now given by

$$\hat{r}'^{\mu_1\mu_2}(p, q) = \varepsilon^{\mu_1\mu_2\alpha\beta} p_\alpha q_\beta \hat{r}'(p, q), \tag{273}$$

$$\begin{aligned}
\hat{r}'(p, q) &= \frac{m c_V^2 c_\pi}{(2\pi)^5 \sqrt{N}} \left[-\Theta(-p^0) \Theta(p^2 - 4m^2) \log_1 \right. \\
&\left. -\Theta(-q^0) \Theta(q^2 - 4m^2) \log_2 -\Theta(-P^0) \Theta(P^2 - 4m^2) \log_3 \right], \tag{274}
\end{aligned}$$

where

$$\log_1 = \log\left(\frac{q^2 + pq + \sqrt{(1 - 4m^2/p^2)N} + i0}{q^2 + pq - \sqrt{(1 - 4m^2/p^2)N} + i0}\right), \tag{275}$$

$$\log_2 = \log\left(\frac{p^2 + pq + \sqrt{(1 - 4m^2/q^2)N} + i0}{p^2 + pq - \sqrt{(1 - 4m^2/q^2)N} + i0}\right), \tag{276}$$

$$\log_3 = \log\left(\frac{-pq + \sqrt{(1 - 4m^2/P^2)N} - i0}{-pq - \sqrt{(1 - 4m^2/P^2)N} - i0}\right). \quad (277)$$

The expressions for a' and d are similar:

$$\hat{a}'^{\mu_1\mu_2} = \varepsilon^{\mu_1\mu_2\alpha\beta} p_\alpha q_\beta \hat{a}'(p, q), \quad (278)$$

$$\hat{a}'(p, q) = \frac{mc_V^2 c_\pi}{(2\pi)^5 \sqrt{N}} \left[-\Theta(p^0) \Theta(p^2 - 4m^2) \log_1 \right. \\ \left. -\Theta(q^0) \Theta(q^2 - 4m^2) \log_2 - \Theta(P^0) \Theta(P^2 - 4m^2) \log_3 \right], \quad (279)$$

$$\hat{d}^{\mu_1\mu_2} = \varepsilon^{\mu_1\mu_2\alpha\beta} p_\alpha q_\beta \hat{d}(p, q), \quad (280)$$

$$\hat{d}(p, q) = \frac{mc_V^2 c_\pi}{(2\pi)^5 \sqrt{N}} \left[\text{sgn}(p^0) \Theta(p^2 - 4m^2) \log_1 \right. \\ \left. + \text{sgn}(q^0) \Theta(q^2 - 4m^2) \log_2 + \text{sgn}(P^0) \Theta(P^2 - 4m^2) \log_3 \right]. \quad (281)$$

Since the scaling limit in Eq. (281) is equal to

$$\lim_{\lambda \rightarrow \infty} \hat{d}^{\mu_1\mu_2}(\lambda p, \lambda q) = \hat{d}_{m=0}^{\mu_1\mu_2}(p, q),$$

we conclude that $\omega_\pi = 0$. However, the central splitting solution is independent of choosing $\omega = 1$ or $\omega = 0$, respectively. To see this, we calculate the difference

$$\hat{r}_{\omega=1}^{\mu_1\mu_2} - \hat{r}_{\omega=0}^{\mu_1\mu_2} = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\hat{d}^{\mu_1\mu_2}(tp, tq)}{1 - t + i0} \left(\frac{1}{t^2} - \frac{1}{t} \right) \\ = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\hat{d}^{\mu_1\mu_2}(tp, tq)}{t^2} = 0,$$

because the denominator is an odd function of t .

To calculate the anomaly, we now insert Eq. (280) into Eq. (263)

$$\hat{a}^{\mu_1\mu_2}(p, q) = \varepsilon^{\mu_1\mu_2\alpha\beta} p_\alpha q_\beta a(p, q), \quad (282)$$

where

$$a(p, q) = \frac{im}{\pi} c_V^2 c_A \int_{-\infty}^{\infty} \frac{dt}{t} \hat{d}(tp, tq), \quad (283)$$

for all $p, q \in V^+$. In Eq. (281) we introduce

$$f_i(p^2, q^2, P^2) = \frac{m}{(2\pi)^5 \sqrt{N}} \log_i, \quad i = 1, 2, 3, \quad (284)$$

and combine the integrals from $-\infty$ to 0 and from 0 to ∞ , taking the sign- functions into account. Substituting $t^2 = \tau$ we get

$$a(p, q) = \frac{i}{\pi} c_V^2 c_A m \int_0^{\infty} \frac{d\tau}{\tau} \left[\Theta(\tau p^2 - 4m^2) f_1(\tau p^2, \tau q^2, \tau P^2) \right.$$

$$+\Theta(\tau q^2 - 4m^2)f_2(\tau p^2, \tau q^2, \tau P^2) + \Theta(\tau P^2 - 4m^2)f_3(\tau p^2, \tau q^2, \tau P^2)\Big]. \quad (285)$$

Since the anomaly is a polynomial of degree $\omega_B + 1 = 2$, $a(p, q)$ must be a pure number independent of p, q , we can take the limit $p^2 \rightarrow 0$ and $q^2 \rightarrow 0$ in Eq. (285), while keeping $P^2 > 0$. Then only the last term contributes. Substituting $\tau P^2 = s$, we obtain

$$a(p, q) = \frac{i}{\pi} c_V^2 c_A m \int_{4m^2}^{\infty} \frac{ds}{s} f_3(0, 0, s). \quad (286)$$

We have for $P^2 \geq 4m^2$:

$$f_3(p^2 = 0, q^2 = 0, P^2) = \frac{2m}{(2\pi)^5 P^2} \log \frac{1 - \sqrt{1 - 4m^2/P^2}}{1 + \sqrt{1 - 4m^2/P^2}},$$

which implies

$$a(p, q) = i \frac{4m^2}{(2\pi)^6} c_V^2 c_A \int_{4m^2}^{\infty} \frac{ds}{s^2} \log \frac{1 - \sqrt{1 - 4m^2/s}}{1 + \sqrt{1 - 4m^2/s}}.$$

Substituting $x = 4m^2/s$, we get

$$a(p, q) = \frac{2i}{(2\pi)^6} c_V^2 c_A \int_0^1 dx \log \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}},$$

which shows the mass independence of the anomaly. The further substitution $\sqrt{1 - x} = z$ makes the integral elementary and we get

$$a(p, q) = -\frac{2i}{(2\pi)^6} c_V^2 c_A. \quad (287)$$

Summing up, according to Eq. (282) the axial anomaly for the triangle graphs is equal to

$$a^{\mu_1 \mu_2}(p, q) = -\frac{2i}{(2\pi)^6} c_V^2 c_A \varepsilon^{\mu_1 \mu_2 \alpha \beta} p_\alpha q_\beta. \quad (288)$$

We have still to investigate whether there exist other splitting solutions which do not have an anomaly while preserving all desired properties of the theory. $\hat{t}^{\mu_1 \mu_2}(p, q)$ is a pseudotensor of rank two. The lowest order normalization polynomial with this property is $\sim \varepsilon^{\mu_1 \mu_2 \alpha \beta} p_\alpha q_\beta$. But this has already $\omega = 2$, in contrast to $\omega_\pi = 0$. Hence, renormalization of the $VV\Pi$ triangle does not help. There seems to be a better chance with $\hat{t}^{\mu_1 \mu_2 \mu_3}(p, q)$, which is a pseudotensor of rank three with $\omega_B = 1$. The most general normalization polynomial which preserves unitarity and the symmetry in the two V vertices is now given by

$$P^{\mu_1 \mu_2 \mu_3}(p, q) = C \varepsilon^{\mu_1 \mu_2 \mu_3 \alpha} (p_\alpha - q_\alpha),$$

where C is a real constant. But this would destroy vector gauge invariance

$$p_{\mu_1} P^{\mu_1 \mu_2 \mu_3}(p, q) = -C \varepsilon^{\mu_1 \mu_2 \mu_3 \alpha} p_{\mu_1} q_\alpha \neq 0,$$

which we do not allow for. That means that the axial anomaly cannot be removed by renormalization. We have to live with it. In the electroweak theory the anomalies cancel by compensation between leptons and quarks ([6], sect. 4.9).

The analysis of the axial anomaly presented above may appear technical at first sight for a reader which is not yet familiar with the causal method. However, working with divergent Feynman integrals has some *ad hoc* character, and since the causal method works without divergent, i.e. ill-defined expressions, the computation of the axial anomaly presented above is more rigorous than in other approaches and serves as an unambiguous consistency check.

5.2 Schwinger Model

The Schwinger model [15, 16] is a popular laboratory for quantum field theoretical methods. As a soluble quantum field theoretical model, its nonperturbative properties and relations to confinement [17, 18] have always been of greatest interest. It is also possible to discuss the model perturbatively in a straightforward way. The interesting features of the model, originally designed to describe QED with massless fermions in 1+1-dimensional spacetime, are related to the fact that the massless fermions and the photon field actually disappear from the physical spectrum, whereas a "physical" massive scalar field appears with the so-called Schwinger mass $m_s^2 = e^2/\pi$, which depends on the coupling constant e . For a full discussion of the model we refer to the literature (see [19] and references therein). In the following, we focus on the calculation of the vacuum polarization (VP) diagram at second order, where the appearance of a mass term in the Schwinger model can be traced in the photon propagator by resummation. It turns out that the correct treatment of the VP is a delicate task, where the careful discussion of the scaling behavior of distributions given above becomes very useful.

5.2.1 The Causal Approach

As previously discussed, the S -matrix is constructed inductively order by order as an operator valued functional in the 1+1-dimensional case

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^2x_1 \dots d^2x_n T_n(x_1, \dots, x_n) g(x_1) \dots g(x_n), \quad (289)$$

where $g(x)$ is a tempered test function that switches the interaction. The first order interaction term for QED given in terms of asymptotic free fields is

$$T_1(x) = ie : \bar{\Psi}(x) \gamma^\mu \Psi(x) : A_\mu(x). \quad (290)$$

We note here that the so-called adiabatic limit $g(x) \rightarrow 1$ has been shown to exist in purely massive theories at each order of the perturbative expansion of the S -matrix [10]. We therefore keep a mass term for the fermion fields in the following, and consider the limit $m \rightarrow 0$ when appropriate. We further note that, of course, the properties of 1+1-dimensional fermion fields do not have much in common with the corresponding counterparts in 3+1-dimensional spacetime, both from a physical and mathematical point of view.

The interesting second order distribution T_2 is constructed by first considering the causal distribution $D_2(x, y)$

$$D_2(x, y) = [T_1(x), T_1(y)] \quad , \quad (291)$$

$$\text{supp } D_2 = \{(x - y) \mid (x - y)^2 \geq 0\} \quad , \quad (292)$$

which has causal support. Then D_2 is split into a retarded and an advanced part $D_2 = R_2 - A_2$, with

$$\text{supp } R_2 = \{(x - y) \mid (x - y)^2 \geq 0, (x^0 - y^0) \geq 0\} \quad , \quad (293)$$

$$\text{supp } A_2 = \{(x - y) \mid (x - y)^2 \geq 0, -(x^0 - y^0) \geq 0\} \quad . \quad (294)$$

Finally T_2 is given by

$$T_2(x, y) = R_2(x, y) + T_1(y)T_1(x) = A_2(x, y) - T_1(x)T_1(y) \quad . \quad (295)$$

For the massive Schwinger model with fermion mass m , the part in the Wick ordered distribution D_2 corresponding to VP

$$D_2(x, y) = e^2 [d_2^{\mu\nu}(x - y) - d_2^{\nu\mu}(y - x)] : A_\mu(x) A_\nu(y) : + \dots \quad (296)$$

then becomes after a short calculation

$$\begin{aligned}\hat{d}_2^{\mu\nu}(k) &:= \frac{1}{2\pi} \int d^2z [d_2^{\mu\nu}(z) - d_2^{\mu\nu}(-z)]e^{ikz}, \\ \hat{d}_2^{\mu\nu}(k) &= \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{4m^2}{2\pi} \frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \text{sgn}(k^0) \Theta(k^2 - 4m^2) \quad .\end{aligned}\quad (297)$$

Obviously, $\hat{d}_2^{\mu\nu}$ has a naive power counting degree $\omega_p = -2$ [20]. But the singular order of the distribution is $\omega = 0$ [21]. Indeed, applying the definitions from sect. 2.1 to $\hat{d}_2^{\mu\nu}(k)$, we obtain the quasi-asymptotics

$$\lim_{\delta \rightarrow 0} \hat{d}_2^{\mu\nu}(k/\delta) = \frac{1}{2\pi} \left(g^{\mu\nu} k^2 - k^\mu k^\nu\right) \delta(k^2) \text{sgn}(k^0) \quad , \quad (298)$$

and we have $\rho(\delta) = 1$, hence $\omega = 0$. The quasi-asymptotics *differs* from the naively expected formal result

$$\hat{d}_2^{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{4m^2}{2\pi} \frac{\Theta(k^2) \text{sgn}(k^0)}{k^2}, \quad (299)$$

which would be ill-defined as a distribution in 2 dimensions. Note that the $g^{\mu\nu}$ -term in Eq. (298) does not contribute to the quasi-asymptotics. The reason for the result Eq. (298) can be explained by the existence of a *sum rule* [22]

$$\int_{4m^2\delta^2}^{\infty} d(q^2) \frac{\delta^2 m^2}{q^4 \sqrt{1 - \frac{4m^2\delta^2}{q^2}}} = \frac{1}{2} \quad , \quad (300)$$

so that the l.h.s. of Eq. (298) is weakly convergent to the r.h.s. In spite of $\text{sgn}(k^0)$, the r.h.s. of Eq. (298) is a well-defined tempered distribution due to the factor $(g^{\mu\nu} k^2 - k^\mu k^\nu)$.

This has the following consequence: The (Fourier transformed) retarded part $r_2^{\mu\nu}$ of $d_2^{\mu\nu}$ would be given in the case $\omega < 0$ by the unsubtracted splitting formula

$$\begin{aligned}\hat{r}_2^{\mu\nu}(k) &= \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{1-t+i0} d_2^{\mu\nu}(tk) \\ &= \frac{im^2}{\pi^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \log \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1} \quad , \quad k^2 > 4m^2, k^0 > 0.\end{aligned}\quad (301)$$

This distribution will vanish in the limit $m \rightarrow 0$, and the photon would remain massless. But since we have $\omega = 0$, we must use the subtracted splitting formula

$$\begin{aligned}\hat{r}_2^{\mu\nu}(k) &= \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{(t-i0)^{\omega+1} (1-t+i0)} \hat{d}_2^{\mu\nu}(tk) \\ &= \frac{im^2}{\pi^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \left(\frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \log \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1} + \frac{1}{2m^2} \right) \quad , \quad k^2 > 4m^2, k^0 > 0.\end{aligned}\quad (302)$$

The new local term survives in the limit $m \rightarrow 0$. After a consistent resummation of the second order VP diagrams, which we will not discuss here, the well-known Schwinger mass term $m_s^2 = e^2/\pi$ appears. Consequently, the difference between simple power-counting and the correct determination of the singular order is by no means a mathematical detail, it is crucial for the proper description of the dynamics of the model. The singular order $\omega = 0$ of the distribution $\hat{d}_2^{\mu\nu}$ implies that a local renormalization is admissible, and even necessary to preserve the gauge structure of the theory.

An further important property of Eq. (302) is its behavior for $k^2 \rightarrow 0$. From $\hat{r}_2^{\mu\nu}$ one obtains the corresponding VP amplitude $\hat{t}_2^{\mu\nu}$ by the replacement $m^2 \rightarrow m^2 - i0$ for arbitrary k . It is then straightforward to show that the logarithmic term in Eq. (302) behaves for $0 > k^2 \rightarrow 0$ like

$$\log \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1} \sim \log \frac{1 + \sqrt{-k^2/4m^2}}{1 - \sqrt{-k^2/4m^2}} \sim \sqrt{-k^2/m^2}, \quad (303)$$

where $k^2 < 0$ and $\log(1 + z) = z$ for $|z| \ll 1$ was used and the case $0 < k^2 \rightarrow 0$ behaves accordingly. Therefore, we obtain the VP amplitude

$$\hat{t}_2^{\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \hat{t}_2(k), \quad (304)$$

where $\hat{t}_2 = \hat{t}_{2,\mu}{}^\mu$ with $\hat{t}_2(k) \rightarrow 0$ for $k^2 \rightarrow 0$. Of course, this observation is a direct consequence of the central splitting solution used in Eq. (302).

The causal method provides the most unambiguous guide to the construction of every order of the perturbative S -matrix.

5.2.2 Dimensional Regularization

We start with the traditional expression for the VP in 1+1 dimensions given by

$$\tilde{t}_{\mu\nu}(k) = \int \frac{d^2 p}{(2\pi)^2} \text{tr} \gamma_\mu \frac{1}{\not{p} - m} \gamma_\nu \frac{1}{\not{p} - \not{k} - m}, \quad (305)$$

where m^2 is used synonymously for $m^2 - i0$. According to the recipes of the dimensional regularization procedure, we consider the trace of Eq. (305) in n dimensions

$$\tilde{t}_\mu{}^\mu(k) = -2^{n/2} \int \frac{d^n p}{(2\pi)^n} \frac{(2-n)(p^2 - pk) + nm^2}{(p^2 - m^2)[(p-k)^2 - m^2]}. \quad (306)$$

We will now proceed in two different ways. First, we perform a naive dimensional regularization by taking the limit $(2-n) \rightarrow 0$ in Eq. (306) before performing the integral. Then only the term

$$\tilde{t}_\mu{}^\mu(k) = 2^{n/2} i \int \frac{d^n p}{(2\pi)^n} \frac{nm^2}{(p^2 - m^2)[(p-k)^2 - m^2]} \quad (307)$$

remains. Inserting the Feynman parameter integral

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \quad (308)$$

and using the 't Hooft-Veltman formula [13]

$$I_0 := \int \frac{d^n p}{(p^2 - 2pk - m^2)^\alpha} = i^{1-2\alpha} \pi^{n/2} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} \frac{1}{(k^2 + m^2)^{\alpha - n/2}} \quad (309)$$

for $\alpha = 2$ and $n = 2$, we obtain

$$I_0 := -\frac{i\pi}{x(1-x)k^2 - m^2} \quad (310)$$

and hence

$$\tilde{t}_\mu{}^\mu(k) = \frac{i}{\pi} \int_0^1 \frac{m^2 dx}{x(1-x)k^2 - m^2}. \quad (311)$$

The important observation is that obviously

$$\tilde{t}_\mu^\mu(k) \rightarrow -i/\pi \quad \text{for } k \rightarrow 0, \quad (312)$$

i.e. we are left with the same problem as in Eq. (301) that Eq. (312) reproduces only "half" the VP amplitude. Note that the result differs by a factor 2π from the result derived in our causal framework, since there a symmetric definition of the (inverse) Fourier transform has been used.

We now renormalize Eq. (305) properly according to the prescriptions of 't Hooft and Veltmann and include all terms in the integral. The full tensor structure is given by

$$\tilde{t}_{\mu\nu}(k) = -2^{n/2} \int \frac{d^n p}{(2\pi)^n} \int_0^1 \frac{[(p^\alpha p^\beta - p^\alpha k^\beta)(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha}) + m^2 g_{\mu\nu}]}{[p^2 - 2pk(1-x) + k^2(1-x) - m^2]^2}. \quad (313)$$

To perform the integrals, we use the 't Hooft-Veltman integrals

$$\int \frac{d^n p p^\mu}{(p^2 - 2pk - m^2)^\alpha} = I_0 k^\mu, \quad (314)$$

$$\int \frac{d^n p p^\mu p^\nu}{(p^2 - 2pk - m^2)^\alpha} = \left(k^\mu k^\nu + \frac{k^2 + m^2}{n + 2 - 2\alpha} g^{\mu\nu} \right) I_0, \quad (315)$$

for $\alpha = 2$ and $n \rightarrow 2$ and obtain

$$\begin{aligned} \tilde{t}_{\mu\nu}(k) = & \frac{2^{n/2} i\pi}{(2\pi)^n} \int_0^1 \frac{dx}{x(1-x)k^2 - m^2} \{ m^2 g_{\mu\nu} - (1-x)k^\alpha k^\beta (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha}) \\ & + [(1-x)^2 k^\alpha k^\beta + \frac{1}{2-n} (x(1-x)k^2 - m^2) g^{\alpha\beta}] (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha}) \}. \end{aligned} \quad (316)$$

Interestingly, the integral above is finite due to a cancellation of the dimensional pole $(2-n)^{-1}$ by a factor generated by

$$g_{\mu\alpha} g_{\nu\beta} - g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha} = (2-n) g_{\mu\nu}, \quad (317)$$

since $g_\mu^\mu = n$. This way we arrive at the *gauge invariant* amplitude

$$\tilde{t}_{\mu\nu}(k) = (k^2 g_{\mu\nu} - k_\mu k_\nu) \tilde{t}(k), \quad (318)$$

$$\tilde{t}(k) = \frac{i}{\pi} \int_0^1 \frac{x(1-x)dx}{x(1-x)k^2 - m^2} \quad (319)$$

with vanishing trace in the limit $k^2 \rightarrow 0$

$$\tilde{t}_\mu^\mu(k) = k^2 \tilde{t}(k) = -\frac{i}{\pi} \frac{k^2}{6m^2} + o\left(\frac{k^4}{m^4}\right) \rightarrow 0. \quad (320)$$

Both results in the dimensional and causal regularization scheme are consistent, however, we observe that the regularization of distributions must be performed with due care.

5.3 Scalar QED in 2+1 Dimensions

In this section we illustrate how gauge invariance is automatically preserved by dimensional regularization by using scalar quantum electrodynamics (sQED) in one time and two space dimensions as an example. As expected, both the causal method and dimensional regularization lead to compatible results, although the underlying premises on which the two methods are based and the resulting perturbative description of the model theory display a rather different behavior.

The traditional starting point of any quantum field theory is a Lagrangean containing coupled classical fields describing the interaction. After quantization, S -matrix elements or Greens functions are constructed with the help of, e.g., Feynman rules. One should accept that this point of view is obsolete to some extent within the causal approach, which is based on a description of the S -matrix by the help of free fields and well-defined interaction terms which can be expressed as a sum of normal-ordered products of free fields. However, the Lagrangean provides a formal tool to express the classical structure of the theory, but one should keep in view that physical theories are always subject to quantization.

The scalar sQED₂₊₁ Lagrangean is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\varphi^\dagger(\partial^\mu + ieA_\mu)\varphi - m^2\varphi^\dagger\varphi, \quad (321)$$

where A_μ describes the electromagnetic field with the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and φ a charged scalar meson field with mass m and electric charge e . The coupling constant e has the dimension of an energy in three-dimensional spacetime, and consequently sQED₂₊₁ is superrenormalizable by naive power counting.

The Lagrangean can be decomposed according to

$$\mathcal{L} = \mathcal{L}_{em}^0 + \mathcal{L}_{matter}^0 + \mathcal{L}_{int}, \quad (322)$$

where the interaction part \mathcal{L}_{int} is given by the minimal coupling of the electromagnetic current to the electromagnetic potential

$$\mathcal{L}_{int} = -j_\mu A^\mu, \quad j_\mu = ie\varphi^\dagger \overleftrightarrow{\partial}_\mu \varphi - e^2\varphi^\dagger\varphi A_\mu. \quad (323)$$

It is now straightforward to construct the Hamiltonian interaction density

$$\mathcal{H}_{int} = ie\varphi^\dagger \overleftrightarrow{\partial}_\mu \varphi A^\mu - e^2\varphi^\dagger\varphi A_\mu A^\mu + e^2\varphi^\dagger\varphi(A^0)^2. \quad (324)$$

Obviously, this expression is not manifestly covariant. It has been shown in [23] that the non-covariant term $-e^2\varphi^\dagger\varphi(A^0)^2$ is canceled in the full perturbative quantum field theory by a local normalization term appearing in the so-called seagull graph at second order in the coupling constant which is sesquilinear in the meson and bilinear in the photon field. This observation applies to scalar sQED in $n + 1$ dimensions in general, and a detailed discussion of sQED₃₊₁ with respect to gauge invariance can be found in [24, 7].

It must also be mentioned that scalar sQED has a pathological infrared behavior both in 3 + 1 and in 2 + 1 dimensions. E.g., the contraction of the two photon lines in the seagull graph with the two photon lines in another seagull graph gives rise to a $\varphi^\dagger\varphi^\dagger\varphi\varphi$ -interaction which is highly singular at short distances, and it is generally accepted that this leads to a transmutation of the original underlying perturbative theory. However, in this work we will focus on strictly perturbative aspects of the theory, which are well-defined at every order of the coupling constant e in our case.

5.3.1 Causal Approach

The crucial difference between the two approaches discussed in this work is the following. In the causal approach, the interaction Hamiltonian density is given by the *normally ordered* product of free quantized

fields, whereas in the dimensional regularization ansatz, all kinds of UV divergences including terms which stem from contractions of fields at the same space-time point are taken into account.

In the Feynman gauge, the free photon field $A_\mu(x)$ fulfills the wave equation

$$\square A_\mu(x) = 0 \quad (325)$$

and the translation invariant distributional commutation relations

$$[A_\mu(x), A_\nu(y)] = [A_\mu(x-y), A_\nu(0)] = ig_{\mu\nu} D_0^{(+)}(x-y), \quad (326)$$

where

$$D_0^{(+)}(x) = \frac{i}{(2\pi)^2} \int d^3p \delta(p^2) \Theta(p^0) e^{-ipx} \quad (327)$$

applies for photonic contractions without time-ordering. For the scalar field, we have the contractions

$$\overline{\varphi(x)} \varphi^\dagger(y) = -i D_m^{(+)}(x-y), \quad (328)$$

$$\overline{\varphi(x)^\dagger} \varphi(y) = +i D_m^{(+)*}(x-y) = +i D_m^{(-)}(y-x) = -i D_m^{(+)}(x-y), \quad (329)$$

where

$$D_m^{(+)}(x) = \frac{i}{(2\pi)^2} \int d^3p \delta(p^2 - m^2) \Theta(p^0) e^{-ipx}. \quad (330)$$

One may choose as a starting point a Hamiltonian density which is given by the normally ordered products of free fields

$$\mathcal{H}_{int}(x) = -(ie\varphi^\dagger(x) \overleftrightarrow{\partial}_\mu \varphi(x) + e^2 \varphi^\dagger(x) \varphi(x) A_\mu(x)) A^\mu(x) \quad (331)$$

and x is an element of 2 + 1-dimensional Minkowski space. The perturbative S -Matrix is then constructed according to the expansion

$$S = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int dx_1^3 \dots dx_n^3 T \{ \mathcal{H}_{int}(x_1) \mathcal{H}_{int}(x_2) \cdot \dots \cdot \mathcal{H}_{int}(x_n) \}, \quad (332)$$

where T is the time-ordering operator.

In the causal approach, we use only the first order term (in the coupling constant) of Eq. (331) (which is motivated by first order interaction term appearing in the corresponding Lagrangian) given by

$$T_1(x) = e : \varphi^\dagger(x) \overleftrightarrow{\partial}_\mu \varphi(x) : A^\mu(x). \quad (333)$$

Thus, the primed distributions can be written as (taking into account that $\widetilde{T}_1 = -T_1$)

$$A'_2(x, y) = -e^2 : \varphi^\dagger(x) \overleftrightarrow{\partial}_\mu \varphi(x) :: \varphi^\dagger(y) \overleftrightarrow{\partial}_\nu \varphi(y) :: A^\mu(x) A^\nu(y) :, \quad (334)$$

$$R'_2(x, y) = -e^2 : \varphi^\dagger(y) \overleftrightarrow{\partial}_\mu \varphi(y) :: \varphi^\dagger(x) \overleftrightarrow{\partial}_\nu \varphi(x) :: A^\mu(x) A^\nu(y) :, \quad (335)$$

the causal distribution is again

$$D_2(x, y) = R'_2(x, y) - A'_2(x, y) = [T_1(x), T_1(y)]. \quad (336)$$

As an example, from Wick's theorem we have

$$\begin{aligned} & : \varphi^\dagger(x) \partial^\mu \varphi(x) :: \varphi^\dagger(y) \partial^\nu \varphi(y) : \\ & =: \varphi^\dagger(x) \partial^\mu \varphi(x) \varphi^\dagger(y) \partial^\nu \varphi(y) : + \varphi^\dagger(x) \partial^\nu \varphi(y) : i \partial_x^\mu D_m^{(+)}(x-y) \end{aligned}$$

$$- : \partial^\mu \varphi(x) \varphi^\dagger(y) : i \partial_y^\nu D_m^{(-)}(y-x) + \partial_x^\mu D_m^{(+)}(x-y) \partial_y^\nu D_m^{(-)}(y-x), \quad (337)$$

with $D_m^{(\pm)}$ the positive/negative frequency part of the Pauli-Jordan distribution.

We focus now on the vacuum polarization diagram, corresponding at second order to

$$D_2(x, y) = -e^2 : A_\mu(x) A_\nu(y) : \quad (338)$$

$$\cdot [\partial_y^\nu D_m^{(+)}(y-x) \partial_x^\mu D_m^{(-)}(x-y) - \partial_y^\nu D_m^{(-)}(y-x) \partial_x^\mu D_m^{(+)}(x-y) \quad (339)$$

$$+ \partial_y^\mu D_m^{(+)}(y-x) \partial_y^\nu D_m^{(-)}(x-y) - \partial_x^\mu D_m^{(-)}(y-x) \partial_y^\nu D_m^{(+)}(x-y) \quad (340)$$

$$- \partial_x^\mu \partial_y^\nu D_m^{(+)}(y-x) D_m^{(-)}(x-y) + \partial_x^\mu \partial_y^\nu D_m^{(-)}(y-x) D_m^{(+)}(x-y) \quad (341)$$

$$- D_m^{(+)}(y-x) \partial_x^\mu \partial_y^\nu D_m^{(-)}(x-y) + D_m^{(-)}(y-x) \partial_x^\mu \partial_y^\nu D_m^{(+)}(x-y)] . \quad (342)$$

For further calculations, we get rid of external fields and change to momentum space

$$D_2(x, y) =: d^{\mu\nu}(x-y) : A_\mu(x) A_\nu(y) : \text{ with } \hat{d}^{\mu\nu}(p) =: \frac{e^2}{4} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \hat{d}(p). \quad (343)$$

In a straightforward manner, one derives ($p = \sqrt{p^2}$)

$$\hat{d}(p) = \frac{p}{2} \Theta(p^2 - 4m^2) \text{sgn}(p_0) \left(1 - \frac{4m^2}{p^2} \right), \quad (344)$$

Obviously, this distribution has singular (and power counting) degree $\omega = 1$.

The retarded distribution $\hat{r}(p)$ follows directly from the splitting formula (first for $p = (p^0, \vec{0})$, $p^0 = p_0 > 0$)

$$\hat{r}(p_0) = i(2\pi)^2 p_0^{\omega+1} \int_{-\infty}^{\infty} dk_0 \frac{|k_0| \Theta(k_0^2 - 4m^2) \text{sgn}(k_0)}{(k_0 - i0)^{\omega+1} (p_0 - k_0 + i0)} \cdot \frac{1}{2} \left(1 - \frac{4m^2}{k_0^2} \right) \quad (345)$$

This dispersion integral can be written

$$\begin{aligned} \hat{r}(p_0) &= 2i\pi^2 p_0^2 \int_{4m^2}^{\infty} \left[\frac{k^0}{k_0^2(p^0 - k^0 + i0)} - \frac{k^0}{k_0^2(p^0 + k^0 + i0)} \right] \left(1 - \frac{4m^2}{k_0^2} \right) dk^0 \\ &= 2i\pi^2 p_0^2 \int_{4m^2}^{\infty} \frac{2k^0 dk^0}{k_0^2((p^0 + i0)^2 - k_0^2)} \left(1 - \frac{4m^2}{k_0^2} \right) \\ &= 2i\pi^2 p_0^2 \int_{4m^2}^{\infty} \frac{ds}{s^{3/2}} \frac{s - 4m^2}{p_0^2 - s + i0p^0}, \end{aligned} \quad (346)$$

where a substitution $s = k_0^2$ was used. For the C-number part r'_2 of R'_2 follows

$$\hat{r}'(p_0) = -\frac{p_0}{2} \Theta(p_0^2 - 4m^2) \Theta(-p_0) \left(1 - \frac{4m^2}{p_0^2} \right) \quad (347)$$

and therefore

$$\hat{t}(p_0) = \hat{r}(p_0) - \hat{r}'(p_0) = 2i\pi^2 p_0^2 \int_{4m^2}^{\infty} \frac{ds}{s^{3/2}} \frac{s - 4m^2}{p_0^2 - s + i0}. \quad (348)$$

The integral can be evaluated by standard methods, and proper analytic continuation of $\hat{r}(p_0)$ leads to the result

$$\hat{t}^{\mu\nu}(p) = -ie^2 \pi^2 p \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left[\frac{2m}{p} + \frac{p^2 - 4m^2}{2p^2} \log \frac{2m + p}{2m - p} \right]. \quad (349)$$

5.3.2 Dimensional Regularization

Applying the well-known Feynman rules to the vacuum polarization diagram leads to

$$\tilde{t}^{\mu\nu} = \int d^3k \frac{2k^\mu p^\nu + 2k^\nu p^\mu - p^\mu p^\nu - 4k^\mu k^\nu}{(k^2 - m^2 + i0) [(k-p)^2 - m^2 + i0]}, \quad (350)$$

using the definition of the Feynman propagator

$$\langle 0 | T [\varphi(x) \varphi^\dagger(y)] | 0 \rangle = -i D_F^m(x-y). \quad (351)$$

The integral can be split into a scalar, vector and tensor part:

$$\tilde{t}^{\mu\nu} = 2p^\nu \mathcal{I}_2^\mu(p) + 2p^\mu \mathcal{I}_2^\nu(p) - p^\mu p^\nu \mathcal{I}_1(p) - 4\mathcal{I}_3^{\mu\nu}(p), \quad (352)$$

with the definitions

$$\mathcal{I}_1(p) := \int d^3k \frac{1}{(k^2 - m^2 + i0) [(k-p)^2 - m^2 + i0]} \quad (353)$$

$$\mathcal{I}_2^\mu(p) := \int d^3k \frac{k^\mu}{(k^2 - m^2 + i0) [(k-p)^2 - m^2 + i0]} \quad (354)$$

$$\mathcal{I}_3^{\mu\nu}(p) := \int d^3k \frac{k^\mu k^\nu}{(k^2 - m^2 + i0) [(k-p)^2 - m^2 + i0]}. \quad (355)$$

We first calculate \mathcal{I}_1 . Using the Feynman parametrization

$$\frac{1}{AB} = \int_0^1 d\alpha [\alpha A + (1-\alpha)B]^{-2} \quad (356)$$

and performing the momentum translation $k^\mu \mapsto k^\mu + \alpha p^\mu$, the integral can be written as

$$\mathcal{I}_1(p) = \int d^3k \int_0^1 d\alpha [k^2 - m^2 + \alpha(1-\alpha)p^2 + i0]^{-2}. \quad (357)$$

Changing the integration dimension to $D = 3 - 2\epsilon$ and using the general relation

$$\int d^D k \frac{(k^2)^r}{(k^2 - a^2 + i0)^m} = i(-1)^{r-m} \pi^{\frac{D}{2}} \frac{\Gamma(r + \frac{D}{2}) \Gamma(m - r - \frac{D}{2})}{\Gamma(\frac{D}{2}) \Gamma(m) (a^2 - i0)^{m-r-\frac{D}{2}}}, \quad (358)$$

the momentum integral can be carried out (with the trivial limit $\epsilon \rightarrow 0$), after which the α -integral becomes simply

$$\mathcal{I}_1(p) = i\pi^2 \int_0^1 \frac{d\alpha}{\sqrt{\alpha(\alpha-1)p^2 + m^2}} = \frac{i\pi^2}{p} \log \frac{2m+p}{2m-p}, \quad (359)$$

where $p = \sqrt{p^2}$.

Note that Eq. (359) is indeed valid for arbitrary p , when m^2 is substituted by $m^2 - i0$. For $p^2 > 4m^2$, the logarithmic term contains both a real and an imaginary part, for $0 < p^2 < 4m^2$, the logarithm is real. For space-like momenta $p^2 < 0$, the logarithm becomes purely imaginary, but also the prefactor $1/p = 1/\sqrt{p^2} = -i/\sqrt{-p^2}$, since the integrand in Eq. (359) is real in this case. For $p^2 < 0$, one can also write

$$\mathcal{I}_1(p) = \frac{i\pi^2}{\sqrt{p^2}} \log \frac{2m + \sqrt{p^2}}{2m - \sqrt{p^2}} = \frac{2i\pi^2}{\sqrt{|p^2|}} \arcsin \left(\frac{1}{1 - 4m^2/p^2} \right), \quad (360)$$

but we will maintain the shorthand used in Eq. (359) in the following.

The same procedure as above can be applied to \mathcal{I}_2^μ , leading to

$$\mathcal{I}_2^\mu(p) = \int d^3k \int_0^1 d\alpha (k^\mu + \alpha p^\mu) [k^2 - m^2 + \alpha(1 - \alpha)p^2 + i0]^{-2}. \quad (361)$$

Integrating symmetrically makes the integral proportional to k^μ disappear, leaving

$$\mathcal{I}_2^\mu(p) = i\pi^2 \int_0^1 d\alpha \frac{\alpha p^\mu}{\sqrt{\alpha(\alpha - 1)p^2 + m^2}} = \frac{i\pi^2 p^\mu}{2p} \log \frac{2m + p}{2m - p}. \quad (362)$$

$\mathcal{I}_3^{\mu\nu}$ Finally, we use Feynman parametrization and momentum translation invariance in order to obtain for $\mathcal{I}_3^{\mu\nu}$

$$\begin{aligned} \mathcal{I}_3^{\mu\nu}(p) &= \int d^3k \int_0^1 d\alpha [k^\mu k^\nu + \alpha(k^\mu p^\nu + p^\mu k^\nu) + \alpha^2 p^\nu p^\nu] \\ &\quad \times [k^2 - m^2 + \alpha(1 - \alpha)p^2 + i0]^{-2}, \end{aligned} \quad (363)$$

where the integrals proportional to an odd power of k vanish.

Rewriting $k^\mu k^\nu = g^{\mu\nu} k^2/D$ and performing $D = 3 - 2\epsilon$ dimensional integration, we arrive at a finite integral for $\epsilon \rightarrow 0$

$$\mathcal{I}_3^{\mu\nu}(p) = i\pi^2 \int_0^1 d\alpha \left[g^{\mu\nu} \sqrt{\alpha(\alpha - 1)p^2 + m^2} + p^\mu p^\nu \frac{\alpha^2}{\sqrt{\alpha(\alpha - 1)p^2 + m^2}} \right], \quad (364)$$

which can be evaluated in a straightforward manner to give

$$\begin{aligned} \mathcal{I}_3^{\mu\nu}(p) &= \frac{i\pi^2}{4} p \left\{ - \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{p} + \frac{p^2 - 4m^2}{2p^2} \log \frac{2m + p}{2m - p} \right] \right. \\ &\quad \left. + g^{\mu\nu} \frac{4m}{p} + \frac{p^\mu p^\nu}{p^2} \log \frac{2m + p}{2m - p} \right\}. \end{aligned}$$

Now, all integrals can be combined in the manner expressed by Eq. (352) to give the full VP

$$\begin{aligned} \tilde{t}^{\mu\nu} &= -i\pi^2 p \cdot \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{p} + \frac{p^2 - 4m^2}{2p^2} \log \frac{2m + p}{2m - p} \right] \\ &\quad + 4i\pi^2 g^{\mu\nu} m. \end{aligned} \quad (365)$$

The interesting observation is given by the fact that the result obtained so far does not exactly match the gauge invariant result derived in the framework of causal perturbation theory. In fact, we have to include the one-loop contracted seagull graph displayed in Fig. (4), which contributes to the photon-photon transition amplitude like the VP diagram displayed in Fig. (3) as well.

Calculating the formal contribution of the fermion-line self-contraction of the seagull graph gives (with the correct normalization factor) by the help of Eq. (358)

$$\tilde{t}_{seagull}^{\mu\nu} = -2g^{\mu\nu} \int \frac{d^3k}{k^2 - m^2 + i0} = -4g^{\mu\nu} i\pi^2 m, \quad (366)$$

which exactly cancels the local term in the scattering matrix element Eq. (365) of the vacuum polarization, which only appears if one uses dimensional regularization.

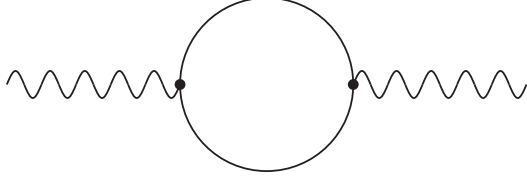


Figure 3: Vacuum polarization

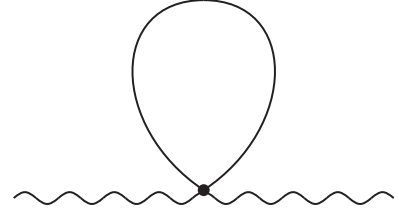


Figure 4: One-loop contracted seagull graph

5.3.3 Pauli-Villars Regularization

\mathcal{I}_1 , according to naive power counting, is convergent and can be calculated using Feynman parameters

$$\mathcal{I}_1(p) = \frac{i\pi^2}{p} \log \frac{2m+p}{2m-p}. \quad (367)$$

For \mathcal{I}_2^μ , we change the propagators according to the procedure

$$\frac{1}{q^2 - m^2 + i0} \mapsto \frac{1}{q^2 - m^2 + i0} - \frac{1}{q^2 - \Lambda^2 + i0}, \quad (368)$$

leading to

$$\mathcal{I}_2^\mu = \int d^3k \frac{k^\mu (m^2 - \Lambda^2)^2}{(k^2 - m^2)(k^2 - \Lambda^2)[(k-p)^2 - m^2][(k-p)^2 - \Lambda^2]}. \quad (369)$$

Subtraction at $p = 0$ gives

$$\begin{aligned} \mathcal{I}_2^\mu(p) &= \mathcal{I}_2^\mu(0) + \tilde{\mathcal{I}}_2^\mu(p), \\ \mathcal{I}_2^\mu(0) &= \int d^3k \frac{k^\mu (m^2 - \Lambda^2)^2}{(k^2 - m^2)^2 (k^2 - \Lambda^2)^2} \\ \tilde{\mathcal{I}}_2^\mu(p) &= \int d^3k \frac{k^\mu (m^2 - \Lambda^2)^2}{(k^2 - m^2)(k^2 - \Lambda^2)} \\ &\quad \left\{ \frac{-1}{(k^2 - m^2)(k^2 - \Lambda^2)} + \frac{1}{[(k-p)^2 - m^2][(k-p)^2 - \Lambda^2]} \right\}. \end{aligned} \quad (370)$$

Taking the limit $\Lambda \rightarrow \infty$ yields

$$\begin{aligned} \mathcal{I}_2^\mu(0) &= 0 \\ \tilde{\mathcal{I}}_2^\mu(p) &= \int d^3k \frac{-k^\mu (p^2 - 2k \cdot p)}{(k^2 - m^2)^2 [(k-p)^2 - m^2]}. \end{aligned}$$

Using Feynman parametrization

$$\frac{1}{ABC} = 2 \int_0^1 d\alpha \int_0^\alpha d\beta [(\alpha - \beta)A + \beta B + (1 - \alpha)C]^{-3} \quad (371)$$

and momentum translation leads to

$$\begin{aligned} \tilde{\mathcal{I}}_2^\mu &= \int d^3k \, 2 \int_0^1 d\alpha \int_0^\alpha d\beta \\ &\times [2p \cdot k \, k^\mu - (1 - 2\beta)p^2 k^\mu + 2\beta p^\mu p \cdot k - (1 - 2\beta)\beta p^2 p^\mu] \\ &\times [k^2 - \beta(\beta - 1)p^2 - m^2 + i0]^{-3} . \end{aligned} \quad (372)$$

Again, the integrals proportional to an odd power of k disappear and we find

$$\tilde{\mathcal{I}}_2^\mu = 2p^\mu \int_0^1 d\alpha \int_0^\alpha d\beta \int d^3k \left[\frac{2}{3}k^2 - (1 - 2\beta)\beta p^2 \right] \cdot [k^2 - \beta(\beta - 1)p^2 - m^2 + i0]^{-3} , \quad (373)$$

which can now be Wick rotated and integrated over k to

$$\begin{aligned} \tilde{\mathcal{I}}_2^\mu &= \frac{i\pi^2}{2} p^\mu \int_0^1 d\alpha \int_0^\alpha d\beta \left\{ \frac{2}{\sqrt{\beta(\beta - 1)p^2 + m^2}} + \frac{(1 - 2\beta)\beta p^2}{[\beta(\beta - 1)p^2 + m^2]^{3/2}} \right\} \\ &= i\pi^2 p^\mu \int_0^1 d\alpha \frac{\alpha}{\sqrt{\alpha(\alpha - 1)p^2 + m^2}} \\ &= \frac{i\pi^2 p^\mu}{2p} \log \frac{2m + p}{2m - p} = \mathcal{I}_2^\mu(p) . \end{aligned} \quad (374)$$

The same procedures may be applied to $\mathcal{I}_3^{\mu\nu}$, too. One has

$$\mathcal{I}_3^{\mu\nu} = \frac{i\pi^2}{4} p \left\{ - \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left[\frac{2m}{p} + \frac{p^2 - 4m^2}{2p^2} \log \frac{2m + p}{2m - p} \right] + \frac{p^\mu p^\nu}{p^2} \log \frac{2m + p}{2m - p} \right\} . \quad (375)$$

Combining all three integrals obtained by Pauli-Villars regularization, one obtains

$$\mathcal{I}^{\mu\nu} = -i\pi^2 p \cdot \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[\frac{2m}{p} + \frac{p^2 - 4m^2}{2p^2} \log \frac{2m + p}{2m - p} \right] , \quad (376)$$

which differs from the solution with dimensional regularization by a term $4i\pi^2 g^{\mu\nu} m$, i.e. gauge invariance is preserved "by hand" by proper normalization of the photon-photon transition amplitude in the present case. However, this is only true if one starts with a normally ordered first order coupling and neglects the one-loop contracted seagull graph from the beginning.

Again, we observe that the calculations differ strongly in different approaches. In the causal method, the non-trivial part of one-loop calculations is a one-dimensional, *finite* dispersion integral. As a general remark, we point out that the causal method, as well as dimensional regularization, has some particular advantages for gauge theories because it does not use a cutoff which breaks gauge invariance, an observation which is illustrated on a rather basic level in this sect. 5.3.3. In a more general setting, it works in a fixed number of physical dimensions so that problems originating from axial couplings, which are also related to the 'γ₅-problem' in dimensional regularization.

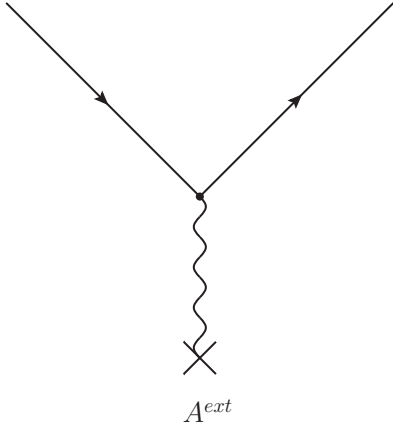


Figure 5: Lowest order scattering

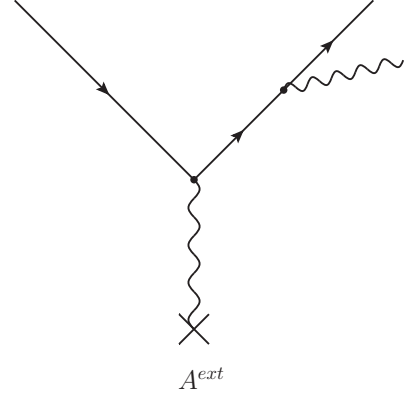


Figure 6: Bremsstrahlung diagram

6 Infrared Divergences

The infrared structure in the causal approach differs strongly from other approaches. Whereas infrared divergences show up as poles in dimensional regularization or as divergences in the artificial mass parameter introduced for the originally massless fields contained in the theory under consideration (a strategy commonly used e.g. in the Pauli-Villars approach), they are automatically regularized by the test function g in Eq. (189). From a mathematical point of view, this is the most natural formulation of the infrared problem, since the T_n 's are operator-valued distributions, and therefore must be smeared out by test functions in $\mathcal{S}(\mathbb{R}^{4n})$, the Schwartz space of functions of rapid decrease. The test function $g \in \mathcal{S}(\mathbb{R}^4)$ plays the role of an "adiabatic switching" and provides a cutoff in the long-range part of the interaction, which can be considered as a natural infrared regulator. An appropriate adiabatic limit $g \rightarrow 1$ must be performed at the end of actual calculations in the right quantities (like cross sections) where this limit exists. Strictly speaking, the perturbative S-matrix according to Eq. (5) does *not* exist for $g \equiv 1$ for many theories involving massless fields. This observation is also closely related to the notion of *infraparticles* [26, 27]. The existence of the adiabatic limit for physical observables is a non-trivial issue of a theory and is not automatically guaranteed at every order of the theory from the mere lowest order definition of the interaction. Below, we will prove the existence of the adiabatic limit for a scattering cross section at fourth order in g for a model theory.

Note that introducing a mass as infrared regularizator for massless fields is a questionable procedure, since it is unclear whether the original massless theory is restored by taking the massless limit of the massive theory, which may suffer from potential problems like, e.g., broken gauge invariance.

In order to demonstrate the causal approach to the infrared problem we consider a theory in 3+1 spacetime dimensions, called totally scalar QED in the following, where a massive scalar charged field is coupled to a massless scalar field, in close analogy to Eq. (216). The corresponding scalar particles will be called meson and photon in the following. The theory is defined by the first order coupling term

$$T_1(x) = -ie : \varphi^\dagger(x)\varphi(x) : [A_0(x) + A^{ext}(x)] = -ie : \varphi^\dagger(x)\varphi(x) : A(x), \quad (377)$$

where $A^{ext}(x)$ denotes an external C-number field and $A_0(x)$ the quantized massless neutral scalar photon field. For dimensional reasons, the coupling constant e has the dimension of an energy or an inverse length.

In the following, we consider the scattering process of the meson off the external field, according to Fig. (5). At first order in the coupling constant, the matrix element for the scattering of a meson with mass m and

initial state momentum \vec{q} and a different final state momentum \vec{p} is given by

$$S_{fi}^{(1)} = \langle f | S^{(1)} | i \rangle = \langle 0 | a(\vec{p}) S^{(1)} a^\dagger(\vec{q}) | 0 \rangle, \quad (378)$$

where $S^{(1)} = -ie \int d^4x_1 : \varphi^\dagger(x_1) \varphi(x_1) : A^{ext}(x_1)$ and $|i\rangle = a^\dagger(\vec{q})|0\rangle$ and $|f\rangle = a^\dagger(\vec{p})|0\rangle$, i.e. a and a^\dagger denote the annihilation and creation operators for one charge type of particles of the meson field, and the vacuum shall be denoted by $|0\rangle$ in this section. One obtains ($k^0 = (k^2 + m^2)^{1/2}$ etc)

$$S_{fi}^{(1)} = -ie \int \frac{d^4x_1}{(2\pi)^3} \int \frac{d^3k d^3k'}{2\sqrt{k_0 k'_0}} e^{-ikx_1} e^{ik'x_1} \langle 0 | a(\vec{p}) : a(\vec{k}) a^\dagger(\vec{k}') : a^\dagger(\vec{q}) | 0 \rangle A^{ext}(x_1). \quad (379)$$

Exploiting the commutation relations and the distributional identity $\int d^n x e^{ikx} = (2\pi)^n \delta(k)$ leads to

$$\begin{aligned} S_{fi} &= -ie \int d^4x_1 \frac{1}{(2\pi)^3} \int \frac{d^3k d^3k'}{2\sqrt{k_0 k'_0}} e^{-ikx_1} e^{ik'x_1} \delta^{(3)}(\vec{q} - \vec{k}) \delta^{(3)}(\vec{k}' - \vec{p}) A^{ext}(x_1) \\ &= \frac{-ie}{(2\pi)^3} \int d^4x_1 \frac{1}{2\sqrt{p_0 q_0}} e^{-iqx_1} e^{ipx_1} A^{ext}(x_1) = \frac{-ie}{2(2\pi)^3 \sqrt{p_0 q_0}} \int d^4x_1 e^{-iqx_1} e^{ipx_1} A^{ext}(x_1) \end{aligned} \quad (380)$$

Introducing the Fourier transform of $A^{ext}(x_1)$, we have

$$\begin{aligned} S_{fi}^{(1)} &= \frac{-ie}{2(2\pi)^3 \sqrt{p_0 q_0}} \int d^4x_1 e^{-iqx_1} e^{ipx_1} \frac{1}{(2\pi)^2} \int d^4k'' e^{-ik''x_1} \hat{A}^{ext}(k'') \\ &= \frac{-ie}{2(2\pi)^3 \sqrt{p_0 q_0}} \frac{1}{(2\pi)^2} \int d^4x_1 \int d^4k'' e^{-iqx_1} e^{ipx_1} e^{-ik''x_1} \hat{A}^{ext}(k'') \\ &= \frac{-ie}{2(2\pi)^3 \sqrt{p_0 q_0}} \frac{1}{(2\pi)^2} \int d^4k'' (2\pi)^4 \delta(-q + p - k'') \hat{A}^{ext}(k''), \end{aligned} \quad (381)$$

and the first order matrix element becomes

$$S_{fi}^{(1)} = \frac{-ie}{2(2\pi) \sqrt{p_0 q_0}} \hat{A}^{ext}(p - q) \quad (382)$$

We assume for the moment that $A^{ext}(x)$ is a Coulomb potential

$$A^{ext}(x) = -\frac{1}{|\vec{x}|}. \quad (383)$$

For the sake of completeness, we calculate the corresponding cross section in detail. In the present case, we have

$$S_{fi}^{(1)} = \frac{ie}{(2\pi)2\sqrt{p_0 q_0}} \left[\frac{1}{(2\pi)^2} \int d^4x \frac{e^{i(p-q)x}}{|\vec{x}|} \right] = \frac{ie}{(2\pi)^3 2\sqrt{p_0 q_0}} \int dx_0 e^{i(p_0 - q_0)x_0} \int d^3x \frac{e^{-i(\vec{p} - \vec{q})\vec{x}}}{|\vec{x}|}. \quad (384)$$

The space integral can be evaluated as follows

$$\begin{aligned} \int d^3x e^{-i(\vec{p} - \vec{q})\vec{x}} \frac{1}{|\vec{x}|} &= -\frac{1}{(\vec{p} - \vec{q})^2} \int d^3x \frac{1}{|\vec{x}|} \Delta e^{-i(\vec{p} - \vec{q})\vec{x}} = -\frac{1}{(\vec{p} - \vec{q})^2} \int d^3x \left(\Delta \frac{1}{|\vec{x}|} \right) e^{-i(\vec{p} - \vec{q})\vec{x}} \\ &= -\frac{1}{(\vec{p} - \vec{q})^2} \int d^3x \left(-4\pi \delta^{(3)}(\vec{x}) \right) e^{-i(\vec{p} - \vec{q})\vec{x}} = \frac{4\pi}{(\vec{p} - \vec{q})^2}, \end{aligned} \quad (385)$$

and thus

$$S_{fi}^{(1)} = \frac{ie}{(2\pi)^3 2\sqrt{p_0 q_0}} (2\pi \delta(p_0 - q_0)) \frac{4\pi}{(\vec{p} - \vec{q})^2}. \quad (386)$$

The transition rate dR from the initial state to a final state within an infinitesimal phase space volume d^3p is given by (T denotes a large time interval)

$$dR = \frac{dW}{T} = \frac{|S_{fi}|^2 d^3p}{T} = \frac{e^2}{(2\pi)^4 p_0 q_0 T} (2\pi\delta(p_0 - q_0))^2 \frac{d^3p}{(\vec{p} - \vec{q})^4}. \quad (387)$$

Of course, the square of the δ -distribution above is ill-defined, since one should work with wave packets in order to get well-defined expressions. However, for the moment we content ourselves with Fermi's trick and perform some *formal* manipulations, starting from

$$2\pi\delta(p_0 - q_0) = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt e^{i(p_0 - q_0)t}. \quad (388)$$

The intuitive argument is that this expression is non-zero for $p_0 = q_0$, so that one may replace one δ -distribution by

$$2\pi\delta(p_0 - q_0) = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \quad (389)$$

and for large T one has $2\pi\delta(p_0 - q_0) = T$. Then dR becomes

$$dR = \frac{e^2}{(2\pi)^4 p_0 q_0} (2\pi\delta(p_0 - q_0)) \frac{d^3p}{(\vec{p} - \vec{q})^4}. \quad (390)$$

The cross section is given by the ratio of the transition rate dR and the flux of incoming particles, given by the initial state expectation value of the operator

$$j^\mu = \frac{i}{2} : \varphi^\dagger(x) \partial^\mu \varphi(x) - \varphi(x) \partial^\mu \varphi^\dagger(x) :, \quad (391)$$

leading to

$$|\vec{j}| = \frac{|\vec{q}|}{(2\pi)^3 q_0} \quad (392)$$

for the chosen initial state. The differential cross section is therefore

$$\begin{aligned} d\sigma &= \frac{dR}{|\vec{j}|} = \frac{e^2}{(2\pi)^4 \left(\frac{|\vec{q}|}{(2\pi)^3 q_0} \right) p_0 q_0 (\vec{p} - \vec{q})^4} (2\pi\delta(p_0 - q_0)) |\vec{p}|^2 d|\vec{p}| d\Omega \\ &= \frac{e^2}{|\vec{q}| p_0 (\vec{p} - \vec{q})^4} \delta(p_0 - q_0) |\vec{p}|^2 d|\vec{p}| d\Omega. \end{aligned} \quad (393)$$

Since $p_0 dp_0 = |\vec{p}| d|\vec{p}|$ and $|\vec{p}| = |\vec{q}|$, we obtain

$$d\sigma = \frac{e^2}{(\vec{p} - \vec{q})^4} \delta(p_0 - q_0) dp_0 d\Omega \quad (394)$$

and finally, performing the integral over p_0 ,

$$\frac{d\sigma}{d\Omega} = \int \frac{e^2}{(\vec{p} - \vec{q})^4} \delta(p_0 - q_0) dp_0 = \frac{e^2}{(\vec{p} - \vec{q})^4} \Big|_{p_0=q_0}, \quad (395)$$

i.e. we basically recover the Rutherford cross section.

6.1 Bremsstrahlung

We now consider the case where the scattered meson emits a soft photon with four-momentum k according to Fig. (6). The scattering matrix at second order is given by

$$S^{(2)} = \frac{(-ie)^2}{2} \int d^4x_1 d^4x_2 : \varphi(x_1) \overline{\varphi^\dagger(x_1) A(x_1)} :: \varphi(x_2) \varphi^\dagger(x_2) A(x_2) : g(x_1) g(x_2) \quad (396)$$

where the contraction symbol denotes one possible Wick contraction of massive fields. Since one has two possibilities to contract the massive fields, the relevant bremsstrahlung term becomes

$$S^{(2)} = (-ie)^2 \int d^4x_1 d^4x_2 (-i) D_F^m(x_1 - x_2) : \varphi(x_1) A(x_1) :: \varphi^\dagger(x_2) A(x_2) : g(x_1) g(x_2) + \dots \quad (397)$$

and the dots denote other terms from the Wick ordering of $S^{(2)}$ which are irrelevant for the present case. The external field operators have the form

$$\varphi(x_1) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k^{(4)}}{\sqrt{2k_0^{(4)}}} [a(\vec{k}^{(4)}) e^{-ik^{(4)}x_1} + b^\dagger(\vec{k}^{(4)}) e^{ik^{(4)}x_1}], \quad (398)$$

$$\varphi^\dagger(x_2) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k'}{\sqrt{2k_0'}} [a^\dagger(\vec{k}') e^{ik'x_2} + b(\vec{k}') e^{-ik'x_2}], \quad (399)$$

$$A(x_1) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k''}{\sqrt{2k_0''}} [c(\vec{k}'') e^{-ik''x_1} + c^\dagger(\vec{k}'') e^{ik''x_1}] + A^{ext}(x_1), \quad (400)$$

$$A(x_2) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k'''}{\sqrt{2k_0'''}} [c(\vec{k}''') e^{-ik'''x_2} + c^\dagger(\vec{k}''') e^{ik'''x_2}] + A^{ext}(x_2), \quad (401)$$

and we obtain the bremsstrahlung matrix element $S_{fi}^{brems} = \langle 0 | a(p) c(k) \hat{S} a^\dagger(q) | 0 \rangle$ after some calculation

$$\begin{aligned} S_{fi}^{brems} &= (-ie)^2 \int d^4x_1 d^4x_2 (-i) D_F^m(x_1 - x_2) \frac{1}{(2\pi)^{(9/2)}} g(x_1) g(x_2) \\ &\left[\int \frac{d^3k^{(4)}}{\sqrt{2k_0^{(4)}}} \frac{d^3k'}{\sqrt{2k_0'}} \frac{d^3k''}{\sqrt{2k_0''}} e^{-ik^{(4)}x_1} e^{ik'x_2} e^{ik''x_1} A^{ext}(x_2) \delta(\vec{p} - \vec{k}') \delta(\vec{q} - \vec{k}^{(4)}) \delta(\vec{k} - \vec{k}'') \right. \\ &\left. + \int \frac{d^3k^{(4)}}{\sqrt{2k_0^{(4)}}} \frac{d^3k'}{\sqrt{2k_0'}} \frac{d^3k'''}{\sqrt{2k_0'''}} e^{-ik^{(4)}x_1} e^{ik'x_2} e^{ik'''x_2} A^{ext}(x_1) \delta(\vec{p} - \vec{k}') \delta(\vec{q} - \vec{k}^{(4)}) \delta(\vec{k} - \vec{k}''') \right] \quad (402) \\ &= \frac{ie^2}{(2\pi)^{9/2}} \int d^4x_1 d^4x_2 D_F^m(x_1 - x_2) g(x_1) g(x_2) \left[\frac{e^{-iqx_1} e^{ipx_2} e^{ikx_1}}{\sqrt{8q_0 p_0 k_0}} A^{ext}(x_2) + \frac{e^{-iqx_1} e^{ipx_2} e^{ikx_2}}{\sqrt{8q_0 p_0 k_0}} A^{ext}(x_1) \right]. \quad (403) \end{aligned}$$

Inserting the Fourier transforms of $g(x_1)$, $g(x_2)$, $A^{ext}(x_1)$, $A^{ext}(x_2)$

$$\begin{aligned} g(x_1) &= \frac{1}{(2\pi)^2} \int d^4k_1 e^{-ik_1x_1} \hat{g}(k_1), \quad g(x_2) = \frac{1}{(2\pi)^2} \int d^4k_2 e^{-ik_2x_2} \hat{g}(k_2), \\ A^{ext}(x_1) &= \frac{1}{(2\pi)^2} \int d^4k_1 e^{-ik_1x_1} \hat{A}^{ext}(k_1), \quad A^{ext}(x_2) = \frac{1}{(2\pi)^2} \int d^4k_2 e^{-ik_2x_2} \hat{A}^{ext}(k_2) \quad (404) \end{aligned}$$

and the Fourier transform of the Feynman propagator leads to

$$S_{fi}^{brems} = \frac{-ie^2}{(2\pi)^{13/2}\sqrt{8q_0p_0k_0}(2\pi)^4(2\pi)^4} \int d^4k_1 d^4k_2 \int d^4k'' d^4k \int d^4x_1 d^4x_2 \left[\frac{e^{-ik''x_1} e^{-iqx_1} e^{ikx_1} e^{-ik_1x_1} e^{ik''x_2} e^{ipx_2} e^{-ik'x_2} e^{-ik_2x_2}}{k''^2 - m^2} A^{ext}(k') \right. \\ \left. \frac{e^{-ik''x_1} e^{-iqx_1} e^{-ik'x_1} e^{-ik_1x_1} e^{ik''x_2} e^{ipx_2} e^{ikx_2} e^{-ik_2x_2}}{k''^2 - m^2} A^{ext}(k') \right] \hat{g}(k_1) \hat{g}(k_2). \quad (405)$$

Note that we sometimes omit the $i0$ -term of the Feynman propagators for the sake of brevity. We evaluate all trivial integrals and arrive at

$$S_{fi}^{brems} = \frac{-ie^2}{2(2\pi)^{13/2}\sqrt{2q_0p_0k_0}} \int d^4k_1 d^4k_2 \left[\frac{1}{(q-k+k_1)^2 - m^2} + \frac{1}{(p+k-k_2)^2 - m^2} \right] A^{ext}(p-q+k-k_1-k_2) \hat{g}(k_1) \hat{g}(k_2). \quad (406)$$

k_1 can be replaced by k_2

$$S_{fi}^{brems} = \frac{-ie^2}{2(2\pi)^{13/2}\sqrt{2q_0p_0k_0}} \int d^4k_1 d^4k_2 \left[\frac{1}{(q-k+k_2)^2 - m^2} + \frac{1}{(p+k-k_2)^2 - m^2} \right] A^{ext}(p-q+k-k_1-k_2) \hat{g}(k_1) \hat{g}(k_2). \quad (407)$$

Now we investigate the adiabatic limit by first replacing $\hat{g}(k_1) \hat{g}(k_2)$ by

$$\frac{1}{\epsilon^4} \hat{g}_0\left(\frac{k_1}{\epsilon}\right) \frac{1}{\epsilon^4} \hat{g}_0\left(\frac{k_2}{\epsilon}\right) \quad (408)$$

corresponding to the replacement $g(x) \rightarrow g_0(\epsilon x)$ in real space, and taking the limit $\epsilon \rightarrow 0$. $g_0(x)$ is a fixed test function in $\mathcal{S}(\mathbb{R}^4)$ with $g_0(0) = 1$, so that $g_0(\epsilon x) \rightarrow 1$ for $\epsilon \rightarrow 0$. Note, however, that $1 \notin \mathcal{S}(\mathbb{R}^4)$. Thus we have

$$S_{fi}^{brems} = \frac{-ie^2}{2(2\pi)^{13/2}\sqrt{q_0p_0}\sqrt{2k_0}} \int \frac{d^4k_1}{\epsilon^4} \frac{d^4k_2}{\epsilon^4} \left[\frac{1}{(q-k+k_2)^2 - m^2} + \frac{1}{(p+k-k_2)^2 - m^2} \right] A^{ext}(p-q+k-k_1-k_2) \hat{g}_0\left(\frac{k_1}{\epsilon}\right) \hat{g}_0\left(\frac{k_2}{\epsilon}\right) \quad (409)$$

or

$$S_{fi}^{brems} = \frac{-ie^2}{2(2\pi)^{13/2}\sqrt{q_0p_0}\sqrt{2k_0}} \int d^4k_1 d^4k_2 \left[\frac{1}{(q-k+\epsilon k_2)^2 - m^2} + \frac{1}{(p+k-\epsilon k_2)^2 - m^2} \right] A^{ext}(p-q+k-\epsilon k_1-\epsilon k_2) \hat{g}_0(k_1) \hat{g}_0(k_2). \quad (410)$$

Envisaging the limit $\epsilon \rightarrow 0$, we can neglect the ϵ -dependent term in the argument of \hat{A}^{ext} and perform one trivial integral. We obtain the result

$$S_{fi}^{brems} = S_{fi}^{(1)} \frac{e}{(2\pi)^{7/2}\sqrt{2k_0}} \int d^4k_2 \left[\frac{1}{2p(k-\epsilon k_2)} - \frac{1}{2q(k-\epsilon k_2)} \right] \hat{g}_0(k_2) \quad (411)$$

where we used the first order matrix element $S_{fi}^{(1)}$. The cross section follows, using the lowest order cross section $\frac{d\sigma^{(1)}}{d\Omega}$

$$\frac{d\sigma(k)}{d\Omega} = \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{(2\pi)^7 2|\vec{k}|} \left| \int d^4k_2 \left[\frac{1}{(-2q(k-\epsilon k_2))} + \frac{1}{2p(k-\epsilon k_2)} \right] \hat{g}_0(k_2) \right|^2$$

$$\begin{aligned}
&= \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{4(2\pi)^7 2|\vec{k}|} \int d^4k_1 d^4k_2 \left[\frac{1}{(q(k - \epsilon k_1))(q(k - \epsilon k_2))} \right. \\
&\quad \left. - \frac{1}{p(k - \epsilon k_1)q(k - \epsilon k_2)} - \frac{1}{q(k - \epsilon k_1)p(k - \epsilon k_2)} + \frac{1}{p(k - \epsilon k_1)p(k - \epsilon k_2)} \right] \hat{g}_0(k_1) \hat{g}_0(k_2). \quad (412)
\end{aligned}$$

For well-known physical reasons, one has to integrate this cross section over the photon momenta up to a cutoff ω_0 , assuming that photons with momenta $< \omega_0$ are not measured. The four integrands can be rewritten by the help of the Feynman trick, leading to

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \int_{|\vec{k}| < \omega_0} d^3k \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{4(2\pi)^7 2|\vec{k}|} \int d^4k_1 d^4k_2 \\
&\quad \left[\int_0^1 dx \frac{1}{[q(k - \epsilon k_1)x + q(k - \epsilon k_2)(1 - x)]^2} - \int_0^1 dx \frac{1}{[p(k - \epsilon k_1)x + q(k - \epsilon k_2)(1 - x)]^2} \right. \\
&\quad \left. - \int_0^1 dx \frac{1}{[q(k - \epsilon k_1)x + p(k - \epsilon k_2)(1 - x)]^2} + \int_0^1 dx \frac{1}{[p(k - \epsilon k_1)x + p(k - \epsilon k_2)(1 - x)]^2} \right] \hat{g}_0(k_1) \hat{g}_0(k_2). \quad (413)
\end{aligned}$$

We first consider the first part of the integral above, the last term can be treated in an analogous manner.

$$\begin{aligned}
I_1 &= \int_0^1 dx \int d^4k_1 d^4k_2 \int_{|\vec{k}| < \omega_0} \frac{d^3k}{2|\vec{k}|} \frac{1}{[q(k - \epsilon k_1)x + q(k - \epsilon k_2)(1 - x)]^2} \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= \int_0^1 dx \int d^4k_1 d^4k_2 \int_{|\vec{k}| < \omega_0} \frac{|\vec{k}| d|\vec{k}| (-d \cos \theta) d\phi}{2[q_0|\vec{k}| - |\vec{q}||\vec{k}| \cos \theta - (q\epsilon k_1 + (1 - x)q\epsilon k_2)]^2} \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= \int_0^1 dx \int d^4k_1 d^4k_2 \int_0^{\omega_0} \frac{|\vec{k}| d|\vec{k}|}{2} \int_{-1}^1 \frac{2\pi d \cos \theta}{[-|\vec{q}||\vec{k}| \cos \theta + q_0|\vec{k}| - (q\epsilon k_1 + (1 - x)q\epsilon k_2)]^2} \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= 2\pi \int_0^1 dx \int d^4k_1 d^4k_2 \int_0^{\omega_0} \frac{d|\vec{k}|}{2|\vec{q}|} \left[\frac{1}{(q_0 - |\vec{q}|)|\vec{k}| - (q\epsilon k_1 + (1 - x)q\epsilon k_2)} \right. \\
&\quad \left. - \frac{1}{(q_0 + |\vec{q}|)|\vec{k}| - (q\epsilon k_1 + (1 - x)q\epsilon k_2)} \right] \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= 2\pi \int_0^1 dx \int d^4k_1 d^4k_2 \frac{1}{2|\vec{q}|} \left[\frac{1}{(q_0 - |\vec{q}|)} \log \left| (q_0 - |\vec{q}|)|\vec{k}| - (q\epsilon k_1 + (1 - x)q\epsilon k_2) \right|_0^{\omega_0} \right. \\
&\quad \left. - \frac{1}{(q_0 + |\vec{q}|)} \log \left| (q_0 + |\vec{q}|)|\vec{k}| - (q\epsilon k_1 + (1 - x)q\epsilon k_2) \right|_0^{\omega_0} \right] \hat{g}_0(k_1) \hat{g}_0(k_2). \quad (414)
\end{aligned}$$

Focusing on the divergent part in the expression above, i.e. examining the logarithms in Eq. (414) for $|\vec{k}| = 0$ only results in

$$I'_1 = 2\pi \int_0^1 dx \int d^4k_1 d^4k_2 \frac{1}{2|\vec{q}|} \left[\frac{1}{(q_0 + |\vec{q}|)} - \frac{1}{(q_0 - |\vec{q}|)} \right] \log \left| (q\epsilon k_1 + (1 - x)q\epsilon k_2) \right| \hat{g}_0(k_1) \hat{g}_0(k_2)$$

$$\begin{aligned}
&= 2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \left(-\frac{1}{q^2} \right) \left[\log |\epsilon| + O(1) \right] \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= -2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \frac{1}{q^2} \log |\epsilon| \hat{g}_0(k_1) \hat{g}_0(k_2) = -\frac{(2\pi)^5}{m^2} \log |\epsilon|, \tag{415}
\end{aligned}$$

such that we obtain the divergent contributions to the cross section from the first and the fourth part in the integral Eq. (413)

$$I'_1 + I'_4 = -\frac{2(2\pi)^5}{m^2} \log |\epsilon|. \tag{416}$$

Finally we calculate the second integral appearing in Eq. (413), the third integral can be calculated analogously. We have

$$\begin{aligned}
I_2 &= -\int_0^1 dx \int d^4 k_1 d^4 k_2 \int_{|\vec{k}| < \omega_0} \frac{d^3 k}{2|\vec{k}|} \frac{1}{[p(k - \epsilon k_1)x + q(k - \epsilon k_2)(1 - x)]^2} \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= -\int_0^1 dx \int d^4 k_1 d^4 k_2 \int_{|\vec{k}| < \omega_0} \frac{|\vec{k}| d|\vec{k}| (-d \cos \theta) d\phi}{2[p(k - \epsilon k_1)x + q(k - \epsilon k_2)(1 - x)]^2} \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= -2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \int_0^{\omega_0} \frac{|\vec{k}| d|\vec{k}|}{2} \int_{-1}^1 \frac{d \cos \theta}{[(x(p - q) + q)k - (xp\epsilon k_1 + (1 - x)q\epsilon k_2)]^2} \hat{g}_0(k_1) \hat{g}_0(k_2). \tag{417}
\end{aligned}$$

We abbreviate $x(p - q) + q$ by Q below.

$$\begin{aligned}
I_2 &= -2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \int_0^{\omega_0} \frac{|\vec{k}| d|\vec{k}|}{2} \int_{-1}^1 \frac{d \cos \theta}{[Q_0 |\vec{k}| - |\vec{Q}| |\vec{k}| \cos \theta - (xp\epsilon k_1 + (1 - x)q\epsilon k_2)]^2} \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= -2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \int_0^{\omega_0} \frac{d|\vec{k}|}{2|\vec{Q}|} \left[\frac{1}{[(Q_0 - |\vec{Q}|) |\vec{k}| - (xp\epsilon k_1 + (1 - x)q\epsilon k_2)]} \right. \\
&\quad \left. - \frac{1}{[(Q_0 + |\vec{Q}|) |\vec{k}| - (xp\epsilon k_1 + (1 - x)q\epsilon k_2)]} \right] \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= -2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \frac{1}{2|\vec{Q}|} \left[\frac{1}{(Q_0 - |\vec{Q}|)} \log \left| (Q_0 - |\vec{Q}|) |\vec{k}| - (xp\epsilon k_1 + (1 - x)q\epsilon k_2) \right|_0^{\omega_0} \right. \\
&\quad \left. - \frac{1}{(Q_0 + |\vec{Q}|)} \log \left| (Q_0 - |\vec{Q}|) |\vec{k}| - (xp\epsilon k_1 + (1 - x)q\epsilon k_2) \right|_0^{\omega_0} \right] \hat{g}_0(k_1) \hat{g}_0(k_2). \tag{418}
\end{aligned}$$

Again, we evaluate only the infrared divergent part of the expression above.

$$I'_2 = -2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \frac{1}{2|\vec{Q}|} \left[\frac{1}{(Q_0 + |\vec{Q}|)} - \frac{1}{(Q_0 - |\vec{Q}|)} \right] \log \left| (xp\epsilon k_1 + (1 - x)q\epsilon k_2) \right| \hat{g}_0(k_1) \hat{g}_0(k_2). \tag{419}$$

Performing the integral over k_1 and k_2 results in

$$\begin{aligned}
I'_2 &= 2\pi \int_0^1 dx \int d^4 k_1 d^4 k_2 \left(\frac{1}{Q^2} \right) \left[\log |\epsilon| + O(1) \right] \hat{g}_0(k_1) \hat{g}_0(k_2) \\
&= (2\pi)^5 \log |\epsilon| \int_0^1 dx \frac{1}{Q^2} = (2\pi)^5 \log |\epsilon| \int_0^1 dx \frac{1}{[x^2(p^2 - 2pq + q^2) + 2x(p - q)q + q^2]^2}. \tag{420}
\end{aligned}$$

Now we use the fact that $pq = m^2 + \frac{1}{2}\vec{P}^2$ for $P = p - q$

$$pq = p_0 q_0 - \vec{p}\vec{q} = E^2 - \vec{p}\vec{q} = m^2 + \frac{\vec{p}^2}{2} + \frac{\vec{q}^2}{2} - \vec{p}\vec{q} = m^2 + \frac{\vec{P}^2}{2}, \tag{421}$$

$$I'_2 = (2\pi)^5 \log |\epsilon| \int_0^1 dx \frac{1}{[-\vec{P}^2 x^2 + \vec{P}^2 x + m^2]^2} = -(2\pi)^5 \log |\epsilon| \frac{1}{\vec{P}^2} \int_0^1 dx \frac{1}{[(x - \frac{1}{2})^2 - (\frac{m^2}{\vec{P}^2} + \frac{1}{4})]^2}. \tag{422}$$

We substitute $y = x - \frac{1}{2}$ and $a = \frac{m^2}{\vec{P}^2} + \frac{1}{4}$ and obtain in a straightforward manner

$$\begin{aligned}
I'_2 &= -(2\pi)^5 \log |\epsilon| \frac{2}{\vec{P}^2} \int_0^{\frac{1}{2}} dy \frac{1}{[y^2 - a]} \\
&= (2\pi)^5 \log |\epsilon| \frac{1}{\frac{|\vec{P}|}{2m} m^2 \sqrt{1 + \frac{\vec{P}^2}{4m^2}}} \log \left| \frac{|\vec{P}|}{2m} + \sqrt{1 + \frac{\vec{P}^2}{4m^2}} \right|. \tag{423}
\end{aligned}$$

For the sake of convenience, we set $b := \frac{|\vec{P}|}{2m}$, so that

$$I'_2 = (2\pi)^5 \log |\epsilon| \frac{1}{m^2 b \sqrt{1 + b^2}} \log \left| b + \sqrt{1 + b^2} \right|. \tag{424}$$

Combining I'_2 with the third integral in Eq. (413), we have

$$I'_2 + I'_3 = (2\pi)^5 \log |\epsilon| \frac{2}{m^2 b \sqrt{1 + b^2}} \log \left| b + \sqrt{1 + b^2} \right|, \tag{425}$$

and the full infrared divergent part of the bremsstrahlung cross section from I'_1 , I'_2 , I'_3 and I'_4 is

$$\frac{d\sigma^{brems}}{d\Omega_{div}} = \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{2(2\pi)^2 m^2} \left[-1 + \frac{1}{b\sqrt{1 + b^2}} \log \left| b + \sqrt{1 + b^2} \right| \right] \log |\epsilon|. \tag{426}$$

6.2 Self Energy

So far we considered the infrared divergences in the bremsstrahlung cross section, which represents a fourth order contribution in the coupling constant e to the inclusive cross section. To see how the logarithmic infrared divergences $\sim \log(\epsilon)$ compensate, we have to investigate the third order Feynman diagrams according to Figs. (7) and (8), since they combine with the first order diagram in Fig. (5) according to

$$\frac{d\sigma}{d\Omega_{4th\ order}} \sim |S_{fi}|_{4th\ order}^2 = |S_{fi}^{(2)}|^2 + 2Re|S_{fi}^{(1)} S_{fi}^{(3)}|. \tag{427}$$

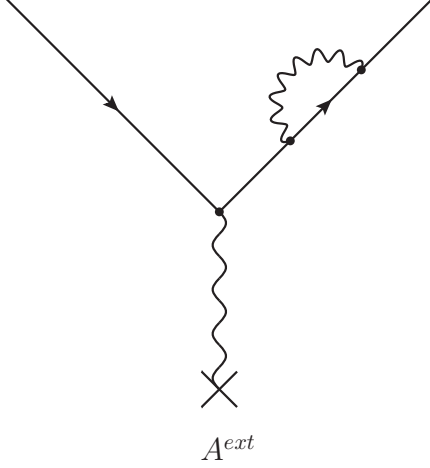


Figure 7: Self energy diagram

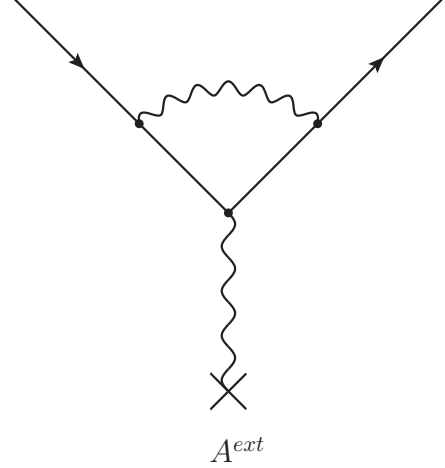


Figure 8: Vertex diagram

We consider first the self energy diagram. The relevant contributions to the S -matrix are given by six equivalent variants of the contraction

$$S^{(3)} = \frac{(-ie)^3}{6} \int d^4x_1 d^4x_2 d^4x_3 g(x_1)g(x_2)g(x_3) \\ : \varphi(x_1)\varphi^\dagger(x_1) \overbrace{A(x_1)} \ :: \varphi(x_2)\varphi^\dagger(x_2) \overbrace{A(x_2)} \ :: \varphi(x_3)\varphi^\dagger(x_3) \overbrace{A(x_3)} \ : . \quad (428)$$

The corresponding third order S -matrix contribution is therefore

$$S^{(3)} = (-ie)^3 \int d^4x_1 d^4x_2 d^4x_3 (-i)D_F^m(x_1 - x_2)(-i)D_F^m(x_2 - x_3)(-i)D_F^0(x_2 - x_3) \\ : \varphi(x_1)\varphi^\dagger(x_3) : A(x_1)g(x_1)g(x_2)g(x_3) + \dots , \quad (429)$$

the massless Feynman propagator D_F^0 coming from the photon propagator. Calculating formally the matrix element with the corresponding initial and final meson states scattering off the external photon field leads to

$$S_{fi}^{self} = \frac{e^3}{2(2\pi)^3 \sqrt{q_0 p_0}} \int d^4k_1 d^4k_2 d^4k_3 \left[\frac{\hat{A}^{ext}(p - q - k_1 - k_2 - k_3)}{(2\pi)^2((p - k_2 - k_3)^2 - m^2)} \right. \\ \left. \int \frac{d^4k}{(2\pi)^6} \frac{1}{k^2((k - (k_3 - p))^2 - m^2)} \right] \hat{g}(k_1)\hat{g}(k_2)\hat{g}(k_3). \quad (430)$$

However, the integral above contains a UV divergent part, which must be handled properly. The formal integral

$$\Sigma^{div}(p^2, k_3, \epsilon) = \int d^4k \frac{1}{k^2((k - (\epsilon k_3 - p))^2 - m^2)} \quad (431)$$

must be regularized or treated within the causal framework. The finite result for the self energy diagram is [28]

$$\Sigma(p^2) = i\pi^2 \left[1 + \frac{m^2 - p^2 - i0}{p^2} \log \left(\frac{m^2 - p^2 - i0}{m^2} \right) \right] + C' , \quad (432)$$

where C' is a free normalization constant, and we must replace the formal integral above by

$$\Sigma((\epsilon k_3 - p)^2) = i\pi^2 \left[1 + \frac{m^2 - (\epsilon k_3 - p)^2}{(\epsilon k_3 - p)^2} \log \left(\frac{m^2 - (\epsilon k_3 - p)^2}{m^2} \right) \right] + C'. \quad (433)$$

In the following, we replace again \hat{g} by \hat{g}_0 and perform all trivial integrals

$$S_{fi}^{brems} = S_{fi}^{(1)} \frac{-ie^2}{(2\pi)^2} \int d^4 k_2 d^4 k_3 \frac{1}{(2\pi)^2} \frac{1}{-2p\epsilon(k_2 + k_3)} \left[\frac{i\pi^2}{(2\pi)^4} \left[1 + \frac{m^2 - (\epsilon k_3 - p)^2}{(\epsilon k_3 - p)^2} \left[\log \left(\frac{m^2 - (\epsilon k_3 - p)^2}{m^2} \right) - i\pi\Theta((\epsilon k_3 - p)^2 - m^2) \right] \right] + C' \right] \hat{g}_0(k_2) \hat{g}_0(k_3). \quad (434)$$

The integral over k_2 gives

$$S_{fi}^{brems} = S_{fi}^{(1)} \frac{-ie^2}{(2\pi)^2} \int d^4 k_3 \frac{1}{-2p\epsilon k_3} \left[\frac{i\pi^2}{(2\pi)^4} \left[1 + \frac{2p\epsilon k_3}{m^2} \left[\log \left(\frac{2p\epsilon k_3}{m^2} \right) - i\pi\Theta(-2p\epsilon k_3) \right] \right] + C' \right] \hat{g}_0(k_3). \quad (435)$$

Now we replace C' by $\frac{i\pi^2}{(2\pi)^4}(C - 1)$

$$\begin{aligned} S_{fi}^{brems} &= S_{fi}^{(1)} \frac{-ie^2}{(2\pi)^2} \int d^4 k_3 \frac{1}{-2p\epsilon k_3} \frac{i\pi^2}{(2\pi)^4} \left[\frac{2p\epsilon k_3}{m^2} \left[\log \left(\frac{2p\epsilon k_3}{m^2} \right) - i\pi\Theta(-2p\epsilon k_3) \right] + C \right] \hat{g}_0(k_3) \\ &= S_{fi}^{(1)} \frac{-e^2}{(2\pi)^2} \int d^4 k_3 \frac{1}{2p\epsilon k_3} \frac{\pi^2}{(2\pi)^4} \left[\frac{2p\epsilon k_3}{m^2} \left[\log |\epsilon| + O(1) \right] + C \right] \hat{g}_0(k_3) \\ &= S_{fi}^{(1)} \frac{-e^2}{(2\pi)^2} \frac{\pi^2}{(2\pi)^4 m^2} \log |\epsilon| \int d^4 k_3 \hat{g}_0(k_3) + S_{fi}^{(1)} \frac{e^2}{(2\pi)^2} \frac{\pi^2}{(2\pi)^4 \epsilon} \int d^4 k_3 \frac{C}{2pk_3} \hat{g}_0(k_3). \end{aligned} \quad (436)$$

The bremsstrahlung diagram contained only logarithmic infrared divergences. For this reason, we choose $C = 0$ in order to avoid a $1/\epsilon$ -divergence in the self energy diagram and obtain

$$S_{fi}^{brems} = S_{fi}^{(1)} \frac{e^2}{(2\pi)^2} \frac{1}{4m^2} \log |\epsilon|, \quad (437)$$

and the corresponding contribution to the cross section follows from

$$\frac{d\sigma}{d\Omega_{4th\ order}} \sim 2Re |S_{fi}^{(1)} S_{fi}^{(3)}| \quad (438)$$

and is given by

$$\frac{d\sigma^{brems}}{d\Omega_{div}} = \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{2(2\pi)^2 m^2} \log |\epsilon|. \quad (439)$$

6.3 Vertex Function

Finally, we consider the vertex diagram according to Fig. (8). Formally, one obtains the expression containing the UV divergent scalar vertex integral

$$\begin{aligned} S_{fi}^{vertex} &= \frac{e^3}{(2\pi)^3 2\sqrt{p_0 q_0}} \hat{A}^{ext}(p - q) \frac{1}{(2\pi)^8} \int d^4 k_1 d^4 k_2 d^4 k_3 \\ &\int d^4 k \frac{1}{((k + q + k_1)^2 - m^2)((k + p - k_3)^2 - m^2)k^2} \hat{g}(k_1) \hat{g}(k_2) \hat{g}(k_3). \end{aligned} \quad (440)$$

Of course, we choose the causal approach to the problem. The third order vertex contribution to the S -matrix can be written

$$S^{(3)} = -e^3 \int d^4x_1 d^4x_2 d^4x_3 : \varphi^\dagger(x_1) t_3^{vertex}(x_1, x_2, x_3) \varphi(x_2) : A(x_3) g(x_1) g(x_2) g(x_3) + \dots \quad (441)$$

The S -matrix element containing the external field is, correspondingly

$$S_{fi}^{vertex} = -e^3 \frac{1}{(2\pi)^3 2\sqrt{p_0 q_0}} \frac{1}{(2\pi)^2} \frac{1}{(2\pi)^6} \int d^4x_1 d^4x_2 d^4x_3 \int d^4k_1 d^4k_2 d^4k_3 \int d^4k'' t_3^{vertex}(x_1, x_2, x_3) e^{ipx_1} e^{-iqx_2} e^{-ik''x_3} e^{-ik_1x_1} e^{-ik_2x_2} e^{-ik_3x_3} \hat{A}^{ext}(k'') \hat{g}(k_1) \hat{g}(k_2) \hat{g}(k_3) \quad (442)$$

yielding

$$S_{fi}^{vertex} = -e^3 \frac{1}{(2\pi)^3 2\sqrt{p_0 q_0}} \int d^4k_1 d^4k_2 d^4k_3 \hat{A}^{ext}(p - q - \epsilon(k_1 + k_2 + k_3)) \left[\frac{1}{(2\pi)^4} \int d^4y_1 d^4y_2 \hat{t}_3^{vertex}(p - \epsilon k_1, q + \epsilon k_2) \right] \hat{g}(\epsilon k_1) \hat{g}(\epsilon k_2) \hat{g}(\epsilon k_3), \quad (443)$$

where (note that $t_3^{vertex}(x_1, x_2, x_3)$ is translation invariant)

$$\hat{t}_3^{vertex}(p, q) = \frac{1}{(2\pi)^4} \int d^4y_1 d^4y_2 t_3^{vertex}(x_1, x_2, x_3) e^{ipy_1 + iqy_2}. \quad (444)$$

Again, one can factor out the first order scattering matrix element, and using the abbreviations $p_1 = p - \epsilon k_1$ and $q_1 = q + \epsilon k_2$ leads to

$$S_{fi}^{vertex} = S_{fi}^{(1)} \frac{-ie^2}{(2\pi)^2} \int d^4k_1 d^4k_2 d^4k_3 \hat{D}(p_1, q_1) \hat{g}(\epsilon k_1) \hat{g}(\epsilon k_2) \hat{g}(\epsilon k_3). \quad (445)$$

In order to calculate \hat{t}_3^{vertex} , we must first construct the causal distribution $\hat{d}_3^{vertex}(p_1, q_1)$, which is given by

$$\hat{d}_3^{vertex}(p_1, q_1) = \frac{1}{(2\pi)^2} \int d^4k \left[\hat{D}_m^{(-)}(k - p_1) \hat{D}_m^{av}(k - q_1) \hat{D}_0^{(+)}(k) - \hat{D}_m^{(-)}(k - q_1) \hat{D}_m^{av}(k - p_1) \hat{D}_0^{(+)}(k) \right. \\ \left. + \hat{D}_m^{(+)}(k - q_1) \hat{D}_m^{ret}(k - p_1) \hat{D}_0^{(-)}(k) - \hat{D}_m^{(+)}(k - p_1) \hat{D}_m^{ret}(k - q_1) \hat{D}_0^{(-)}(k) \right. \\ \left. + \hat{D}_m^{(+)}(k - q_1) \hat{D}_m^{(-)}(k - p_1) \hat{D}_F^0(k) - \hat{D}_m^{(+)}(k - p_1) \hat{D}_m^{(-)}(k - q_1) \hat{D}_F^0(k) \right]. \quad (446)$$

The full calculation of \hat{d}_3^{vertex} can be found in [7, 29]. The result is

$$\hat{d}_3^{vertex}(p_1, q_1) = \frac{\pi}{4(2\pi)^6 \sqrt{N}} \left[\text{sgn}(p_{1_0}) \Theta(p_1^2 - m^2) \log \left| \frac{q_1^2 - m^2 - p_1 q_1 (1 - \frac{m^2}{p_1^2}) + \sqrt{N} (1 - \frac{m^2}{p_1^2})}{q_1^2 - m^2 - p_1 q_1 (1 - \frac{m^2}{p_1^2}) - \sqrt{N} (1 - \frac{m^2}{p_1^2})} \right| \right. \\ \left. - \text{sgn}(q_{1_0}) \Theta(q_1^2 - m^2) \log \left| \frac{p_1^2 - m^2 - p_1 q_1 (1 - \frac{m^2}{q_1^2}) + \sqrt{N} (1 - \frac{m^2}{q_1^2})}{p_1^2 - m^2 - p_1 q_1 (1 - \frac{m^2}{q_1^2}) - \sqrt{N} (1 - \frac{m^2}{q_1^2})} \right| \right]$$

$$\text{sgn}(P_0)\Theta(P^2 - 4m^2) \log \left| \frac{p_1 q_1 + m^2 + \sqrt{N} \sqrt{1 - \frac{4m^2}{P^2}}}{p_1 q_1 + m^2 - \sqrt{N} \sqrt{1 - \frac{4m^2}{P^2}}} \right|. \quad (447)$$

The time-ordered distribution $\hat{t}_3^{\text{vertex}}$ is obtained from $\hat{d}_3^{\text{vertex}}$ by distribution splitting, i.e. from a subtracted dispersion integral according to the vertex scaling degree $\omega = 0$. The infrared divergence is contained in the two first logarithmic terms of Eq. (447), and one may write

$$J_{\text{div}} = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{I_{\text{div}}(tp_1, tq_1)}{t^2(1-t+i0)} \quad (448)$$

with a first term

$$J_{\text{div}_1} = \frac{i}{8\sqrt{N}(2\pi)^6} \int_{-\infty}^{\infty} dt \frac{\text{sgn}(t)\Theta(t^2 p_1^2 - m^2)}{t^2(1-t+i0)} \log \left| \frac{\frac{m^2}{p_1^2} p_1 q_1 - m^2 + t^2(q_1^2 - p_1 q_1) - \frac{m^2}{p_1^2} \sqrt{N} + \sqrt{N} t^2}{\frac{m^2}{p_1^2} p_1 q_1 - m^2 + t^2(q_1^2 - p_1 q_1) + \frac{m^2}{p_1^2} \sqrt{N} - \sqrt{N} t^2} \right|. \quad (449)$$

This integral can be evaluated in a straightforward manner and leads to expressions containing Spence functions and logarithms. We restrict ourselves to the term which contains the infrared divergence

$$\begin{aligned} J_{\text{div}_1} &= \frac{i}{8\sqrt{N}(2\pi)^6} \log \left| \frac{m^2 - p_1^2}{p_1^2} \right| \log \left| \frac{\frac{m^2}{p_1^2} p_1 q_1 - m^2 + q_1^2 - p_1 q_1 + \sqrt{N} \left(1 - \frac{m^2}{p_1^2}\right)}{\frac{m^2}{p_1^2} p_1 q_1 - m^2 + q_1^2 - p_1 q_1 - \sqrt{N} \left(1 - \frac{m^2}{p_1^2}\right)} \right| \\ &= \frac{i}{8\sqrt{N}(2\pi)^6} \log \left| \frac{m^2 - p_1^2}{p_1^2} \right| \log \left| \frac{p_1^2(q_1^2 - m^2) - p_1 q_1(p_1^2 - m^2) + \sqrt{N}(p_1^2 - m^2)}{p_1^2(q_1^2 - m^2) - p_1 q_1(p_1^2 - m^2) - \sqrt{N}(p_1^2 - m^2)} \right| \\ &= -\frac{i}{8\sqrt{N}(2\pi)^6} \log \left| \frac{m^2 - p_1^2}{p_1^2} \right| \log \left| \frac{p_1^2 \frac{(q_1^2 - m^2)}{(p_1^2 - m^2)} - p_1 q_1 - \sqrt{N}}{p_1^2 \frac{(q_1^2 - m^2)}{(p_1^2 - m^2)} - p_1 q_1 + \sqrt{N}} \right|. \end{aligned} \quad (450)$$

Now we use the explicit form of p_1 and q_1

$$p_1 = p - \epsilon k_1 \rightarrow (p_1^2 - m^2) = -2\epsilon p k_1 + O(1), \quad (451)$$

$$q_1 = q + \epsilon k_2 \rightarrow (q_1^2 - m^2) = 2\epsilon q k_2 + O(1), \quad (452)$$

and $\sqrt{N} = m|\vec{P}|\sqrt{1 + \frac{\vec{P}^2}{4m^2}}$, leading to

$$\begin{aligned} J_{\text{div}_1} &= -\frac{i}{(2\pi)^6 8m|\vec{P}|\sqrt{1 + \frac{\vec{P}^2}{4m^2}}} [\log |\epsilon| + O(1)] \log \left| \frac{-\frac{qk_2}{pk_1} m^2 - m^2 - \frac{\vec{P}^2}{2} - \frac{\vec{P}^2}{2} \sqrt{1 + \frac{4m^2}{\vec{P}^2}}}{-\frac{qk_2}{pk_1} m^2 - m^2 - \frac{\vec{P}^2}{2} + \frac{\vec{P}^2}{2} \sqrt{1 + \frac{4m^2}{\vec{P}^2}}} \right| \\ &= -\frac{i}{(2\pi)^6 8m|\vec{P}|\sqrt{1 + \frac{\vec{P}^2}{4m^2}}} [\log |\epsilon| + O(1)] \\ &\quad \log \left| \frac{\left(1 - \frac{qk_2}{pk_1}\right) + \left(1 + \frac{qk_2}{pk_1}\right) \sqrt{1 + \frac{4m^2}{\vec{P}^2}} + \left(1 - \frac{qk_2}{pk_1}\right) \sqrt{1 + \frac{4m^2}{\vec{P}^2}} + \left(1 + \frac{qk_2}{pk_1}\right) \left(1 + \frac{4m^2}{\vec{P}^2}\right)}{\left(1 - \frac{qk_2}{pk_1}\right) - \left(1 + \frac{qk_2}{pk_1}\right) \sqrt{1 + \frac{4m^2}{\vec{P}^2}} - \left(1 - \frac{qk_2}{pk_1}\right) \sqrt{1 + \frac{4m^2}{\vec{P}^2}} + \left(1 + \frac{qk_2}{pk_1}\right) \left(1 + \frac{4m^2}{\vec{P}^2}\right)} \right| \\ &= -\frac{i}{(2\pi)^6 8m|\vec{P}|\sqrt{1 + \frac{\vec{P}^2}{4m^2}}} [\log |\epsilon| + O(1)] \end{aligned}$$

$$\left[\log \left| \frac{1 + \sqrt{1 + \frac{4m^2}{\vec{P}^2}}}{1 - \sqrt{1 + \frac{4m^2}{\vec{P}^2}}} \right| + \log \left| \frac{1 - \frac{4m^2}{\vec{P}^2} + (1 + \frac{qk_2}{pk_1})\sqrt{1 + \frac{4m^2}{\vec{P}^2}}}{1 - \frac{4m^2}{\vec{P}^2} - (1 + \frac{qk_2}{pk_1})\sqrt{1 + \frac{4m^2}{\vec{P}^2}}} \right| \right]. \quad (453)$$

J_{div2} is calculated along the same lines

$$J_{div2} = -\frac{i}{(2\pi)^6 8m |\vec{P}| \sqrt{1 + \frac{\vec{P}^2}{4m^2}}} [\log |\epsilon| + O(1)] \left[\log \left| \frac{1 + \sqrt{1 + \frac{4m^2}{\vec{P}^2}}}{1 - \sqrt{1 + \frac{4m^2}{\vec{P}^2}}} \right| - \log \left| \frac{1 - \frac{4m^2}{\vec{P}^2} + (1 + \frac{qk_1}{pk_3})\sqrt{1 + \frac{4m^2}{\vec{P}^2}}}{1 - \frac{4m^2}{\vec{P}^2} - (1 + \frac{qk_1}{pk_3})\sqrt{1 + \frac{4m^2}{\vec{P}^2}}} \right| \right]. \quad (454)$$

The two results finally combine to

$$J_{div} = -\frac{i}{(2\pi)^6 4m^2 \frac{|\vec{P}|}{2m} \sqrt{1 + \frac{\vec{P}^2}{4m^2}}} \log \left| \frac{|\vec{P}|}{2m} + \sqrt{1 + \frac{\vec{P}^2}{4m^2}} \right| \log |\epsilon| \quad (455)$$

or

$$J_{div} = -\frac{i}{(2\pi)^6 4m^2 b \sqrt{1 + b^2}} \log \left| b + \sqrt{1 + b^2} \right| \log |\epsilon|. \quad (456)$$

The divergent vertex contribution to the cross section follows

$$\frac{d\sigma^{vertex}}{d\Omega_{div}} = -\frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{2(2\pi)^2 m^2 b \sqrt{1 + b^2}} \log \left| b + \sqrt{1 + b^2} \right| \log |\epsilon|. \quad (457)$$

Obviously, the self energy and vertex infrared divergences now cancel the divergence generated by the bremsstrahlung process:

$$\frac{d\sigma^{brems}}{d\Omega_{div}} = \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{2(2\pi)^2 m^2} \left[-1 + \frac{1}{b\sqrt{1 + b^2}} \log \left| b + \sqrt{1 + b^2} \right| \right] \log |\epsilon|, \quad (458)$$

$$\frac{d\sigma^{self}}{d\Omega_{div}} = \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{2(2\pi)^2 m^2} \log |\epsilon|, \quad (459)$$

$$\frac{d\sigma^{vertex}}{d\Omega_{div}} = \frac{d\sigma^{(1)}}{d\Omega} \frac{e^2}{2(2\pi)^2 m^2} \left[-\frac{1}{b\sqrt{1 + b^2}} \log \left| b + \sqrt{1 + b^2} \right| \right] \log |\epsilon|. \quad (460)$$

This shows that the adiabatic limit $g \rightarrow 1$ exists in the causal formalism for the inclusive cross-section. For a discussion of the uniqueness of the adiabatic limit we refer to [7, 30].

We conclude this section by highlighting the qualitative picture of the calculations given above. The switching of the interaction with a test function g , which vanishes for large space and time distances, corresponds to a gedanken experiment where the charged mesons are liberated from their (scalar) "electromagnetic" field. The non-perturbative description of interacting fields is highly non-trivial and a hitherto unsolved problem. The good news is the fact that physical observables can be constructed in our model theory in an unambiguous way in the limit $g \rightarrow 1$, where the interaction becomes permanent.

7 Gauge Theories

7.1 Spin 1

As we have emphasized before all known interactions in nature can be described by quantum gauge theories. Gravity can be described within a very similar causal setting as "ordinary" spin-1 gauge theory, as will be

shown below. However, higher order perturbative quantum gravity holds the highly non-trivial problem of non-renormalizability, which may potentially show up in the causal framework as a violation of perturbative quantum gauge invariance. At least, the theory is still consistent at second order in the gravitational constant and may provide an effective description of the interaction.

By quantum gauge theory we mean a theory which has a gauge invariant S-matrix. This is different from classical gauge invariance where the classical Lagrangian is gauge invariant. Instead we define gauge invariance for the time-ordered products T_n constructed by causal perturbation theory as described in sect. 3 using the gauge variations of the free fields.

One is tempted to define perturbative gauge invariance simply by $d_Q T_n = 0$, but this is not correct. To find the right definition let us consider QED where we certainly know what gauge invariance means. Ordinary spinor quantum electrodynamics is constructed from

$$T_1(x) = ie : \bar{\psi}(x) \gamma^\mu \psi(x) : A_\mu(x). \quad (461)$$

The *free* Dirac fields $\psi, \bar{\psi}$ have zero gauge variation $d_Q \psi = 0 = d_Q \bar{\psi}$, but $d_Q A_\mu = i \partial_\mu u$. Then we obtain

$$d_Q T_1 = -e : \bar{\psi} \gamma^\mu \psi : \partial_\mu u = ie \partial_\mu (i : \bar{\psi} \gamma^\mu \psi : u).$$

Here we have used current conservation

$$\partial_\mu : \bar{\psi} \gamma^\mu \psi := 0$$

which follows from the *free* Dirac equations. We see that $d_Q T_1$ is not zero, but a divergence

$$d_Q T_1 = i \partial_\mu T_{1/1}^\mu, \quad (462)$$

where

$$T_{1/1}^\mu = ie : \bar{\psi} \gamma^\mu \psi : u \quad (463)$$

is called *Q*-vertex in the following and Eq. (462) establishes the first order gauge invariance.

It is not hard to generalize this to higher orders. If we freely interchange d_Q and the time ordering we can write

$$\begin{aligned} d_Q T_n &= d_Q T \{ T_1(x_1) \cdot \dots \cdot T_1(x_n) \} \\ &= \sum_{l=1}^n T \{ T_1(x_1) \dots d_Q T_1(x_l) \dots T_1(x_n) \} \\ &= \sum_{l=1}^n T \{ T_1(x_1) \dots i \partial_\mu T_{1/1}^\mu(x_l) \dots T_1(x_n) \}. \end{aligned} \quad (464)$$

The time ordered products herein have to be constructed correctly by the causal method, using the *Q*-vertex from Eq. (463) at x_l instead of the ordinary QED vertex Eq. (461). Again formally taking the derivative out of the *T*-product we get

$$\begin{aligned} d_Q T_n &= i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} T \{ T_1(x_1) \dots T_{1/1}^\mu(x_l) \dots T_1(x_n) \} \\ &\stackrel{\text{def}}{=} i \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} T_{n/l}^\mu(x_1 \dots x_n). \end{aligned} \quad (465)$$

This equation certainly holds for $x_j \neq x_k$, for all $j \neq k$, because there we can calculate with the *T*-product in the same way as with an ordinary product. But the extension to the diagonal $x_1 = \dots = x_n$ produces local terms in general, both in Eq. (464) and in Eq. (465). If it is possible to absorb such local terms by suitable normalization of the distributions T_n and $T_{n/l}^\mu$, then we call the theory gauge invariant to *n*-th order. We want

to emphasize that perturbative gauge invariance not only means that $d_Q T_n$ is a divergence, the divergence must also be of the specific form Eq. (465) involving the Q -vertex.

Now we check what perturbative gauge invariance defined by Eq. (465) means for the total S-matrix. Applying the gauge variation d_Q to the formal power series we obtain

$$d_Q S(g) = \sum_{n=1}^{\infty} \frac{i}{n!} \int d^4 x_1 \dots d^4 x_n \sum_{l=1}^n (\partial_{\mu}^{x_l} T_{n/l}^{\mu}) g(x_1) \dots g(x_n).$$

Since the test function $g(x)$ is in Schwartz space, we can integrate by parts

$$= - \sum_{n=1}^{\infty} \frac{i}{n!} \int d^4 x_1 \dots d^4 x_n \sum_{l=1}^n T_{n/l}^{\mu} g(x_1) \dots (\partial_{\mu} g)(x_l) \dots g(x_n).$$

If it is possible to perform the so-called adiabatic limit $g \rightarrow 1$ here, then the right-hand side goes to zero and we get the naive definition of gauge invariance of the S-matrix

$$\lim_{g \rightarrow 1} d_Q S(g) = 0. \quad (466)$$

The adiabatic limit exists if all gauge fields are massive. It does not exist for the time-ordered products if some gauge field is massless. In this case Eq. (466) is meaningless and we must use the perturbative definition in Eq. (465). The latter is really at the heart of gauge theory because it determines the possible couplings T_1 . This we are now going to show for massless spin-1 gauge fields.

We consider a collection of vector fields $A_a^{\mu}(x)$, $a = 1, \dots, N$ and ghost fields $u_a(x)$ with anti-ghost fields $\tilde{u}_a(x)$ quantized in the usual manner according to Eq. (156). It is our goal to find all possible gauge invariant self-couplings $T_1(x)$ of these fields. Since the gauge variation d_Q generates ghost fields from vector fields, it is pretty clear that T_1 must involve ghost and anti-ghost fields as well. But we assume ghost number = 0, so that u and \tilde{u} must appear in pairs. It is sufficient to consider trilinear couplings proportional to a coupling constant g , quadrilinear ones proportional to g^2 correspond to T_2 and should come out automatically in the causal construction. We therefore start from the following general ansatz [31]

$$T_1(x) = ig \{ f_{abc}^1 : A_{\mu a} A_{\nu b} \partial^{\nu} A_c^{\mu} : + f_{abc}^2 : A_{\mu a} A_b^{\mu} \partial^{\nu} A_c^{\nu} : \\ + f_{abc}^3 : A_{\mu a} u_b \partial^{\mu} \tilde{u}_c + f_{abc}^4 : (\partial^{\mu} A_{\mu a}) u_b \tilde{u}_c : + f_{abc}^5 : A_{\mu a} (\partial^{\mu} u_b) \tilde{u}_c : \}. \quad (467)$$

Here we have further assumed that T_1 is a Lorentz scalar, and for the sake of simplicity we only consider CP conserving terms here. T_1 being a Lorentz scalar, we need an odd number of derivatives in each term. We only consider one derivative because with three the theory is not renormalizable. The f_{abc}^j are arbitrary constants, but unitarity requires a skew-adjoint T_1

$$T_1^{\dagger}(x) = -T_1(x),$$

so that the f 's and g must be real. This was the reason for the imaginary i in Eq. (467). Since the Wick monomial in the second term is symmetric in a and b , we assume

$$f_{abc}^2 = f_{bac}^2$$

without loss of generality. The reader easily convinces himself that there is no further possibility to contract the Lorentz indices and place the derivative. All double indices including a, b, c are summed over.

Next we calculate the gauge variation

$$d_Q T_1 = - \left\{ f_{abc}^1 (\partial_{\mu} u_a A_{\nu b} \partial^{\nu} A_c^{\mu} + A_{\mu a} \partial_{\nu} u_b \partial^{\nu} A_c^{\mu} + A_{\mu a} A_{\nu b} \partial^{\nu} \partial^{\mu} u_c) \right. \\ \left. + f_{abc}^2 (2 \partial_{\mu} u_a A_b^{\mu} \partial_{\nu} A_c^{\nu} + A_{\mu a} A_b^{\mu} \partial_{\nu} \partial^{\nu} u_c) \right.$$

$$\begin{aligned}
& + f_{abc}^3 (\partial_\mu u_a u_b \partial^\mu \tilde{u}_c + A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu) \\
& + f_{abc}^4 (\partial^\mu \partial_\mu u_a u_b \tilde{u}_c + (\partial^\mu A_{\mu a}) u_b \partial_\nu A_c^\nu) \\
& + f_{abc}^5 (\partial_\mu u_a \partial^\mu u_b \tilde{u}_c + A_{\mu a} \partial^\mu u_b \partial_\nu A_c^\nu) \Big\}. \tag{468}
\end{aligned}$$

The last term in the second and the first one in the fourth line vanish due to the wave equation. To simplify the notation we do not write the double dots for normal ordering anymore, all products of field operators with the same argument are normally ordered *if nothing else is said*.

For gauge invariance the expression Eq. (468) must be a divergence

$$= i \partial_\mu T_{1/1}^\mu(x).$$

We therefore write down a general ansatz for $T_{1/1}^\mu$ as well:

$$\begin{aligned}
iT_{1/1}^\mu = g \Big\{ & g_{abc}^1 \partial^\mu u_a A_{\nu b} A_c^\nu + g_{abc}^2 u_a A_{\nu b} \partial^\mu A_c^\nu \\
& + g_{abc}^3 \partial_\nu u_a A_b^\nu A_c^\mu + g_{abc}^4 u_a \partial_\nu A_b^\nu A_c^\mu \\
& + g_{abc}^5 u_a A_b^\nu \partial_\nu A_c^\mu + g_{abc}^6 u_a u_b \partial^\mu \tilde{u}_c + g_{abc}^7 \partial^\mu u_a u_b \tilde{u}_c \Big\}. \tag{469}
\end{aligned}$$

The symmetry in the first and antisymmetry in the sixth term give the relations

$$g_{abc}^1 = g_{acb}^1, \quad g_{abc}^6 = -g_{bac}^6.$$

This ansatz for $T_{1/1}^\mu$ can be further restricted using the nilpotence property

$$id_Q \partial_\mu T_{1/1}^\mu = d_Q^2 T_1 = 0.$$

Substituting Eq. (469) and collecting terms with the same field operators we obtain the following homogeneous relations:

$$\partial^\mu u_a \partial_\mu \partial_\nu u_b A_c^\nu : \quad 2g_{abc}^1 + g_{acb}^2 + g_{abc}^3 - g_{bac}^3 - g_{bca}^3 + g_{acb}^5 = 0 \tag{470}$$

$$\partial^\mu u_a \partial_\mu A_b^\nu \partial_\nu u_c : \quad 2g_{abc}^1 + g_{acb}^2 - g_{cba}^3 - g_{cab}^5 = 0 \tag{471}$$

$$\partial_\mu u_a \partial^\mu u_b \partial_\nu A_c^\nu : \quad g_{abc}^3 - g_{bac}^3 + g_{acb}^4 - g_{bac}^4 + g_{abc}^7 - g_{bac}^7 = 0 \tag{472}$$

$$u_a \partial^\mu u_b \partial_\mu \partial_\nu A_c^\nu : \quad g_{abc}^4 + g_{abc}^5 + g_{abc}^6 - g_{bac}^6 - g_{bac}^7 = 0 \tag{473}$$

$$u_a \partial^\mu \partial_\nu u_b \partial_\mu A_c^\nu : \quad g_{abc}^2 + g_{acb}^2 + g_{abc}^5 + g_{acb}^5 = 0. \tag{474}$$

First order gauge invariance according to Eq. (468) now implies linear relations between the f 's and g 's:

$$\partial^\mu u_a \partial_\mu A_{\nu b} A_c^\nu : \quad -f_{cab}^1 = 2g_{abc}^1 + g_{acb}^2 \tag{475}$$

$$\partial^\mu u_a A_{\mu b} \partial_\nu A_c^\nu : \quad -2f_{abc}^2 - f_{bac}^5 = g_{abc}^3 + g_{acb}^4 \tag{476}$$

$$u_a \partial_\nu A_b^\nu \partial_\mu A_c^\mu : \quad -f_{bac}^4 - f_{cab}^4 = g_{abc}^4 + g_{acb}^4 \tag{477}$$

$$\partial_\mu u_a u_b \partial^\mu \tilde{u}_c : \quad -f_{abc}^3 = 2g_{abc}^6 + g_{abc}^7 \tag{478}$$

$$\partial_\mu u_a \partial^\mu u_b \tilde{u}_c : \quad -f_{abc}^5 + f_{bac}^5 = g_{abc}^7 - g_{bac}^7 \tag{479}$$

$$\partial_\mu \partial_\nu u_a A_b^\nu A_c^\mu : \quad -f_{cba}^1 - f_{bca}^1 = g_{abc}^3 + g_{acb}^3 \tag{480}$$

$$\partial_\mu u_a \partial_\nu A_b^\nu A_c^\mu : \quad -f_{acb}^1 = g_{abc}^3 + g_{acb}^5 \tag{481}$$

$$u_a \partial_\nu A_b^\nu \partial_\mu A_c^\nu : \quad g_{abc}^2 = -g_{acb}^2 \tag{482}$$

$$u_a \partial_\mu \partial_\nu A_b^\mu A_c^\nu : \quad -f_{cab}^3 = g_{acb}^4 + g_{abc}^5 \quad (483)$$

$$u_a \partial_\nu A_b^\mu \partial_\mu A_c^\nu : \quad g_{abc}^5 = -g_{acb}^5. \quad (484)$$

All information comes out of this linear system. Since the elimination process is somewhat tedious, we give all details to save the readers time. Let us interchange b and c in Eq. (481)

$$-f_{abc}^1 = g_{acb}^3 + g_{acb}^5 \quad (485)$$

and add this to Eq. (481)

$$-f_{abc}^1 - f_{acb}^1 = g_{abc}^3 + g_{acb}^3 + g_{abc}^5 + g_{acb}^5. \quad (486)$$

By Eq. (484) g^5 drops out and by Eq. (480) the right side is equal to

$$= -f_{cba}^1 - f_{bca}^1. \quad (487)$$

This implies

$$f_{abc}^1 - f_{cba}^1 = f_{bca}^1 - f_{acb}^1. \quad (488)$$

Let us now decompose f_{abc}^1 into symmetric and antisymmetric parts in the first and third indices:

$$f_{abc}^1 = d_{abc} + f_{abc}, \quad d_{abc} = d_{cba}, \quad f_{abc} = -f_{cba}, \quad (489)$$

then Eq. (488) implies

$$f_{abc} = -f_{acb} = -f_{cba} = f_{cab} = f_{bca} = -f_{bac}. \quad (490)$$

So we arrive at the important result that f_{abc} is totally antisymmetric. The Jacobi identity follows from second order gauge invariance, hence, f_{abc} can be regarded as structure constants of a *real* Lie algebra.

The total antisymmetry of f_{abc} implies the total symmetry of d_{abc} . Next we use the representation Eq. (489) in Eq. (475):

$$-f_{cab} - d_{cab} = 2g_{abc}^1 + g_{abc}^2. \quad (491)$$

Here g_{abc}^2 is antisymmetric in b, c according to Eq. (482), so that g_{abc}^1 must be symmetric, hence

$$g_{abc}^1 = -\frac{1}{2}d_{cab} = -\frac{1}{2}d_{abc} \quad (492)$$

$$g_{abc}^2 = f_{cab} = f_{abc}. \quad (493)$$

Now we can write Eqns. (480) and (481) in the form

$$g_{abc}^3 + g_{acb}^3 = -2d_{cba} = -2d_{abc} \quad (494)$$

$$-f_{acb} - d_{acb} = g_{abc}^3 + g_{acb}^5. \quad (495)$$

Since g_{abc}^5 is antisymmetric in b, c according to Eq. (484), the symmetric part of this equation agrees with Eq. (494) and the antisymmetric part is given by

$$-f_{acb} = \frac{1}{2}(g_{abc}^3 - g_{acb}^3) + g_{acb}^5. \quad (496)$$

Hence, we find

$$g_{abc}^5 = g_{abc}^3 - f_{abc} + d_{abc}, \quad (497)$$

where Eq. (494) has been taken into account.

Now we turn to Eq. (473) and substitute g^5 from Eq. (497)

$$g_{abc}^4 = g_{cab}^7 - 2g_{acb}^6 - (g_{acb}^3 - f_{acb} + d_{abc}). \quad (498)$$

Using this in Eq. (472) we see that g^3 and g^7 cancel out so that finally

$$g_{abc}^6 = \frac{1}{2}f_{abc}. \quad (499)$$

Then Eq. (498) can be simplified to

$$g_{abc}^4 = g_{cab}^7 - g_{acb}^3 - d_{abc}. \quad (500)$$

Substituting this into Eq. (476) gives

$$f_{abc}^2 = -\frac{1}{2}(f_{bac}^5 + g_{bac}^7 - d_{abc}). \quad (501)$$

f^3 follows from Eq. (478):

$$f_{abc}^3 = -2g_{abc}^6 - g_{abc}^7 = -g_{abc}^7 - f_{abc}. \quad (502)$$

On the other hand, from Eq. (483) we get a different result

$$f_{abc}^3 = -g_{cba}^7 + f_{abc}, \quad (503)$$

which implies

$$g_{abc}^7 = g_{cba}^7 - 2f_{abc}. \quad (504)$$

Finally, from Eq. (477) we conclude

$$f_{bac}^4 + g_{bac}^7 = -f_{cab}^4 - g_{cab}^7 \quad (505)$$

and Eq. (479) gives another symmetry relation

$$f_{abc}^5 + g_{abc}^7 = f_{bac}^5 + g_{bac}^7. \quad (506)$$

It is easily checked that with the results just obtained all equations are identically satisfied.

Summing up we have obtained the following form of the trilinear coupling

$$\begin{aligned} T_1 = ig \left\{ (f_{abc} + d_{abc}) A_{\mu a} A_{\nu b} \partial^\nu A_c^\mu - \frac{1}{2}(f_{bac}^5 + g_{bac}^7 - d_{abc}) A_{\mu a} A_b^\mu \partial_\nu A_c^\nu \right. \\ \left. - (g_{abc}^7 + f_{abc}) A_{\mu a} u_b \partial^\mu \tilde{u}_c + f_{abc}^4 (\partial^\mu A_{\mu a}) u_b \tilde{u}_c + f_{abc}^5 A_{\mu a} (\partial^\mu u_b) \tilde{u}_c \right\}. \end{aligned} \quad (507)$$

The terms proportional to d_{abc} give a divergence

$$d_{abc} (A_{\mu a} A_{\nu b} \partial^\nu A_c^\mu + \frac{1}{2} A_{\mu a} A_b^\mu \partial_\nu A_c^\nu) = \frac{1}{2} d_{abc} \partial^\nu (A_{\mu a} A_b^\mu A_{\nu c}). \quad (508)$$

This can be left out because it does not change the S-matrix. Next it is important to remember the relation Eq. (505) which shows the antisymmetry with respect to b and c . Therefore, we have

$$C_1 \stackrel{\text{def}}{=} ig (f_{abc}^4 + g_{abc}^7) (\partial^\mu A_{\mu a}) u_b \tilde{u}_c = -\frac{1}{2} g (f_{abc}^4 + g_{abc}^7) d_Q (\tilde{u}_a u_b \tilde{u}_c). \quad (509)$$

Such a term which is d_Q of "something" is called a coboundary in cohomology theory [32]. Using Eq. (509) in Eq. (508) we have to add the term with g^7 which is taken into account as follows

$$\begin{aligned} & -g_{abc}^7 (A_{\mu a} u_b \partial^\mu \tilde{u}_c + \partial^\mu A_{\mu a} u_b \tilde{u}_c) \\ & = -g_{abc}^7 \partial^\mu (A_{\mu a} u_b \tilde{u}_c) + g_{abc}^7 A_{\mu a} \partial^\mu u_b \tilde{u}_c. \end{aligned}$$

Now T_1 assumes the following form

$$T_1 = ig \left\{ f_{abc} A_{\mu a} A_{\nu b} \partial^\nu A_c^\mu - \frac{1}{2} (f_{abc}^5 + g_{abc}^7) A_{\mu a} A_b^\mu \partial^\nu A_c^\nu \right.$$

$$\begin{aligned}
& -f_{abc}A_{\mu a}u_b\partial^\mu\tilde{u}_c + (f_{abc}^5 + g_{abc}^7)A_{\mu a}\partial^\mu u_b\tilde{u}_c \Big\} \\
& + \frac{i}{2}gd_{abc}\partial^\nu(A_{\mu a}A_b^\mu A_{\nu c}) - igg_{abc}^7\partial^\mu(A_{\mu a}u_b\tilde{u}_c) + C_1.
\end{aligned} \tag{510}$$

Due to Eq. (506) the second and fourth term together give a second coboundary

$$\begin{aligned}
C_2 &= ig(f_{abc}^5 + g_{abc}^7)(A_{\mu a}\partial^\mu u_b\tilde{u}_c - \frac{1}{2}A_{\mu a}A_b^\mu\partial_\nu A_\nu^c) \\
&= \frac{i}{2}g(f_{abc}^5 + g_{abc}^7)d_Q(A_{\mu a}A_b^\mu\tilde{u}_c).
\end{aligned} \tag{511}$$

The coboundary terms lead to an equivalent S-matrix as well.

Omitting the trivial divergence and coboundary terms we arrive at the following final result

$$T_1 = igf_{abc}(A_{\mu a}A_{\nu b}\partial^\nu A_c^\mu - A_{\mu a}u_b\partial^\mu\tilde{u}_c). \tag{512}$$

This is the well-known Yang-Mills plus ghost coupling to lowest order. At second order, gauge invariance gives the remaining coupling terms of pure Yang-Mills theory (see [6], sect. 3.4).

The real strength of the method comes out in massive gauge theories. Since in S-matrix theory the asymptotic free fields are the basic objects, one has to start with massive gauge fields from the beginning. Then gauge invariance of first and second order has to work and fixes all couplings. In particular, a physical scalar field, the Higgs field is necessary to satisfy second order gauge invariance. But the Brout-Englert-Higgs mechanism and spontaneous symmetry breaking plays no immediate role in such an approach. For details we refer to [6].

In order to motivate the formal accomplishments constructed so far, we conclude by giving a rather qualitative comparison of the present formalism to the textbook literature. Above, we observed that QED is gauge invariant, but the true importance of gauge invariance is the fact that it allows to prove on a formal level the *unitarity* of the S-matrix on the physical subspace [41]. The presence of a skew-adjoint operator A^0 in the first order interaction or the presence of unphysical longitudinal and timelike photon states causes the S-matrix to be non-unitary on the full Fock space, but it is on the physical subspace. In QED, ghosts are introduced only as a formal tool, since they 'blow up' the Fock space unnecessarily, and they do not interact with the electrons and photons. But in QCD, the situation is not so trivial, due to the self-coupling of the gauge fields.

The gluon vector potential can be represented by the traceless Hermitian 3×3 standard Gell-Mann matrices λ^a , $a = 1, \dots, 8$

$$A_\mu = \sum_{a=1}^8 A_\mu^a \frac{\lambda^a}{2} =: A_\mu^a \frac{\lambda^a}{2}. \tag{513}$$

The λ 's satisfy the commutation and normalization relations

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2} \right] = if_{abc} \frac{\lambda^c}{2}, \quad \text{tr}(\lambda^a \lambda^b) = 2\delta_{ab}, \tag{514}$$

and the numerical values of the structure constants $f_{abc} = -f_{bac} = -f_{acb}$ can be found in numerous QCD textbooks. Since we are working with a fixed matrix representation, we do not care whether the color indices are upper or lower indices.

The natural generalization of the QED Lagrangian to the Lagrangian of purely gluonic QCD is

$$\mathcal{L}_{gluon} = -\frac{1}{2}\text{tr} G_{\mu\nu} G^{\mu\nu} = -\frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu}, \tag{515}$$

with

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{516}$$

or, using the first relation of Eq. (514)

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc}A_\mu^b A_\nu^c. \tag{517}$$

It is an important detail that we are working with *interacting* classical fields here, therefore the corresponding field strength tensor G contains a term proportional to the coupling constant in contrast to the free fields $F_{\mu\nu}^{free} = \partial_\mu A_\nu^{free} - \partial_\nu A_\mu^{free}$ used throughout this paper. \mathcal{L}_{gluon} is invariant under classical local gauge transformations

$$A_\mu(x) \rightarrow U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g}U(x)\partial_\mu U^{-1}(x), \quad (518)$$

where $U(x) \in SU(3)$.

We extract now the first order gluon coupling from the Lagrangian. The Lagrangian

$$\mathcal{L}_{gluon} = -\frac{1}{4}[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc}A_\mu^b A_\nu^c][\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + gf_{ab'c'}A_{b'}^\mu A_{c'}^\nu], \quad (519)$$

contains obviously the free field part (this terminology is not really correct, since we are dealing with interacting fields here)

$$\mathcal{L}_{gluon}^{free} = -\frac{1}{4}[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a][\partial^\mu A_a^\nu - \partial^\nu A_a^\mu] \quad (520)$$

and the first order interaction part is given by

$$\mathcal{L}_{gluon}^{int} = -\frac{1}{4}[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a][gf_{ab'c'}A_{b'}^\mu A_{c'}^\nu] - \frac{1}{4}[gf_{abc}A_\mu^b A_\nu^c][\partial^\mu A_a^\nu - \partial^\nu A_a^\mu] \quad (521)$$

$$= -\frac{g}{2}f_{abc}A_\mu^b A_\nu^c[\partial^\mu A_a^\nu - \partial^\nu A_a^\mu] = -\frac{g}{2}f_{abc}A_\mu^a A_\nu^b[\partial^\mu A_c^\nu - \partial^\nu A_c^\mu] \quad (522)$$

$$= gf_{abc}A_\mu^a A_\nu^b \partial^\nu A_c^\mu. \quad (523)$$

The first interaction terms comes out from classical symmetry considerations here; in the framework presented in this paper, it is the consequence of purely quantum mechanical considerations.

Since we are working in Feynman gauge, we add the corresponding gauge fixing term \mathcal{L}_{gf} to the Lagrangian. Additionally, we add the ghost term which describes the ghost interaction. The total Lagrangian then reads

$$\mathcal{L}_{QCD} = \mathcal{L}_{gluon} + \mathcal{L}_{gf} + \mathcal{L}_{ghost} \quad (524)$$

$$= \mathcal{L}_{gluon} - \frac{1}{2}(\partial_\mu A_a^\mu)^2 + \partial^\mu \tilde{u}(\partial_\mu u_a - gf_{abc}u_b A_{\mu c}). \quad (525)$$

The classical ghosts are anticommuting Grassmann numbers, i.e. $u^2 = \tilde{u}^2 = 0$, $u\tilde{u} = -\tilde{u}u$.

The BRST transformation is defined by

$$\delta A_\mu^a = i\lambda(\partial_\mu u_a - gf_{abc}u_b A_{\mu c}), \quad (526)$$

$$\delta \tilde{u}_a = -i\lambda\partial_\mu A_a^\mu, \quad (527)$$

$$\delta u_a = \frac{g}{2}\lambda f_{abc}u_b u_c, \quad (528)$$

where λ is a space-time independent anticommuting Grassmann variable. The special property of the BRST transformation is the fact that the actions

$$S_{gluon} = \int d^4x \mathcal{L}_{gluon}, \quad S_{gf} + S_{ghost} = \int d^4x (\mathcal{L}_{gf} + \mathcal{L}_{ghost}) \quad (529)$$

and $S_{total} = S_{gluon} + S_{gf} + S_{ghost}$ are all invariant under the transformation:

$$\delta S_{gluon} = 0, \quad \delta(S_{gf} + S_{ghost}) = 0. \quad (530)$$

The similarity of free quantum gauge transformation introduced in this paper to the BRST transformation is obvious. One important difference is the absence of interaction terms $\sim g$. Furthermore, the free quantum gauge transformation is a transformation of free quantum fields, whereas the BRST transformation is a transformation of classical fields, which enter in path integrals when the theory is quantized. Finally, the free gauge transformation leaves the T_n 's invariant up to divergences, whereas the BRST transformation is a symmetry of the full QCD Lagrangian. How the two symmetries are intertwined perturbatively is explained in [35]. A more rigorous axiomatic approach is discussed in [36, 37].

7.2 Spin 2

The crucial test of the gauge principle is spin 2 where it should lead to a quantum theory of gravity. In this case we supplement the gauge invariance condition

$$[Q, T(x)] = d_Q T(x) = i\partial_\alpha T^\alpha(x), \quad (531)$$

in the following way. Since d_Q and the space-time derivative ∂_α commute it follows from nilpotency that

$$\partial_\alpha d_Q T^\alpha = 0. \quad (532)$$

If the appropriate form of the Poincaré lemma is true, this implies

$$d_Q T^\alpha = [Q, T^\alpha] = i\partial_\beta T^{\alpha\beta} \quad (533)$$

with antisymmetric $T^{\alpha\beta}$. In the same way we get

$$[Q, T^{\alpha\beta}] = i\partial_\gamma T^{\alpha\beta\gamma} \dots \quad (534)$$

with totally antisymmetric $T^{\alpha\beta\gamma}$ and so on. These are the so-called descent equations (similar to Wess-Zumino consistency conditions). It is our aim to find a solution of these equations describing the self-coupling of the symmetric tensor field $h^{\mu\nu}$ considered in sect. 3.1, Eq. (176). We recall the gauge variations Eq. (180):

$$d_Q h^{\mu\nu} = -\frac{i}{2}(\partial^\nu u^\mu + \partial^\mu u^\nu - \eta^{\mu\nu} \partial_\alpha u^\alpha) \quad (535)$$

$$d_Q u^\mu = 0$$

$$d_Q \tilde{u}^\mu = i\partial_\nu h^{\mu\nu}, \quad (536)$$

where we now denote the Minkowski tensor by $\eta^{\mu\nu}$ to distinguish it from Einstein's $g^{\mu\nu}$.

The descent procedure starts from $T^{\alpha\beta\gamma}$ which must contain three ghost fields u and two derivatives and is totally antisymmetric. To exclude trivial couplings we require that it does not contain a co-boundary $d_Q B$ for some $B \neq 0$. Then there are the following two possibilities only:

$$\partial^\beta u^\alpha u^\mu \partial_\mu u^\gamma, \quad \partial^\alpha u^\mu \partial_\mu u^\beta u^\gamma. \quad (537)$$

Therefore we start the descent procedure with the expression

$$\begin{aligned} T^{\alpha\beta\gamma} = & a_1(\partial^\beta u^\alpha u^\mu \partial_\mu u^\gamma - \partial^\alpha u^\beta u^\mu \partial_\mu u^\gamma - \partial^\beta u^\gamma u^\mu \partial_\mu u^\alpha - \partial^\gamma u^\alpha u^\mu \partial_\mu u^\beta \\ & + \partial^\alpha u^\gamma u^\mu \partial_\mu u^\beta + \partial^\gamma u^\beta u^\mu \partial_\mu u^\alpha) + a_2(\partial^\alpha u^\mu \partial_\mu u^\beta u^\gamma - \partial^\beta u^\mu \partial_\mu u^\alpha u^\gamma \\ & - \partial^\gamma u^\mu \partial_\mu u^\beta u^\alpha - \partial^\alpha u^\mu \partial_\mu u^\gamma u^\beta + \partial^\gamma u^\mu \partial_\mu u^\alpha u^\beta + \partial^\beta u^\mu \partial_\mu u^\gamma u^\alpha). \end{aligned} \quad (538)$$

Next we have to compute $\partial_\gamma T^{\alpha\beta\gamma}$ and this is equal to $-id_Q T^{\alpha\beta}$ by Eq. (534). To determine $T^{\alpha\beta}$ requires an "integration" d_Q^{-1} . As always in calculus this integration can be achieved by making a suitable ansatz for $T^{\alpha\beta}$ and fixing the free parameters. The following 5 parameter expression will do:

$$\begin{aligned} T^{\alpha\beta} = & b_1 u^\mu \partial_\mu u_\nu \partial^\beta h^{\alpha\nu} + b_2 u^\mu \partial_\nu u^\alpha \partial_\mu h^{\beta\nu} + b_3 u^\alpha \partial_\nu u^\mu \partial_\mu h^{\beta\nu} \\ & + \frac{b_4}{2} \partial_\mu u^\alpha \partial_\nu u^\beta h^{\mu\nu} + b_5 \partial_\mu u^\mu \partial_\nu u^\alpha h^{\beta\nu} - (\alpha \leftrightarrow \beta). \end{aligned} \quad (539)$$

Substituting this into Eq. (534) leads to

$$b_1 = -2a_1, \quad b_2 = -2a_2 = -2a_1, \quad b_3 = 2a_1, \quad b_4 = -4a_1, \quad b_5 = -2a_1.$$

An overall factor is arbitrary, we take $a_1 = -1$ which gives

$$T^{\alpha\beta} = 2(u^\mu \partial_\mu u_\nu \partial^\beta h^{\alpha\nu} + u^\mu \partial_\nu u^\alpha \partial_\mu h^{\beta\nu} - u^\alpha \partial_\nu u^\mu \partial_\mu h^{\beta\nu} + \partial_\mu u^\alpha \partial_\nu u^\beta h^{\mu\nu} + \partial_\nu u^\nu \partial_\mu u^\alpha h^{\beta\mu}) - (\alpha \leftrightarrow \beta). \quad (540)$$

In a similar way we compute $\partial_\beta T^{\alpha\beta}$ and make an ansatz for T^α . The latter now has to contain ghost-antighost couplings also. The precise form can be taken from the following final result:

$$\begin{aligned} T^\alpha = & 4u^\mu \partial_\mu h_{\beta\nu} \partial^\beta h^{\alpha\nu} - 2u^\mu \partial_\mu h^{\beta\nu} \partial^\alpha h_{\beta\nu} - 2u^\alpha \partial^\beta h^{\mu\nu} \partial_\mu h_{\beta\nu} - 4\partial_\nu u_\beta \partial_\mu h^{\alpha\beta} h^{\mu\nu} + 4\partial_\nu u^\nu \partial^\mu h^{\alpha\beta} h_{\beta\mu} \\ & + u^\alpha \partial_\beta h_{\mu\nu} \partial^\beta h^{\mu\nu} - 2\partial_\nu u^\nu h_{\mu\beta} \partial^\alpha h^{\mu\beta} - \frac{1}{2} u^\alpha \partial_\mu h \partial^\mu h + \partial_\nu u^\nu h h^\alpha + u^\nu \partial_\nu h \partial^\alpha h - 2\partial_\nu u^\mu h^{\mu\nu} \partial^\alpha h \\ & + 4\partial^\nu u_\mu \partial^\alpha h^{\mu\beta} h_{\beta\nu} - 4\partial^\nu u^\mu \partial_\mu h^{\alpha\beta} h_{\beta\nu} - 2u^\mu \partial_\mu u^\nu \partial^\alpha \tilde{u}_\nu + 2u^\mu \partial_\nu u^\alpha \partial_\mu \tilde{u}^\nu \\ & - 2u^\alpha \partial_\nu u^\mu \partial_\mu \tilde{u}^\nu + 2\partial_\nu u^\nu \partial_\mu u^\alpha \tilde{u}^\mu + 2u^\mu \partial_\mu \partial_\nu u^\nu \tilde{u}^\alpha - 2u^\alpha \partial_\mu \partial_\nu u^\mu \tilde{u}^\nu. \end{aligned} \quad (541)$$

The last step calculating $\partial_\alpha T^\alpha$ and setting it equal to $-id_Q T$ gives the trilinear coupling of massless gravity

$$\begin{aligned} T = & -h^{\alpha\beta} \partial_\alpha h \partial_\beta h + 2h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu} + 4h_{\alpha\beta} \partial_\nu h^{\beta\mu} \partial_\mu h^{\alpha\nu} + 2h_{\alpha\beta} \partial_\mu h^{\alpha\beta} \partial^\mu h - 4h_{\alpha\beta} \partial_\nu h^{\alpha\mu} \partial^\nu h^\beta_\mu \\ & - 4u^\mu \partial_\beta \tilde{u}_\nu \partial_\mu h^{\nu\beta} + 4\partial_\nu u^\beta \partial_\mu \tilde{u}_\beta h^{\mu\nu} - 4\partial_\nu u^\nu \partial_\mu \tilde{u}^\beta h^{\beta\mu} + 4\partial_\nu u^\mu \partial_\mu \tilde{u}_\beta h^{\nu\beta}. \end{aligned} \quad (542)$$

This coupling should have something to do with general relativity. To see this we leave quantum field theory aside and take the metric tensor $g_{\mu\nu}$ as the fundamental classical field. The indices are no longer Lorentz indices, they are raised and lowered with $g^{\nu\rho}$ itself which is defined as the inverse $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$. One also introduces the determinant

$$g = \det g_{\mu\nu}. \quad (543)$$

Our starting point is the Einstein-Hilbert action given by

$$S_{EH} = -\frac{2}{\kappa^2} \int d^4x \sqrt{-g} R, \quad \kappa^2 = 32\pi G, \quad (544)$$

where G is Newton's constant. R is the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} \quad (545)$$

which follows from the Ricci tensor

$$R_{\mu\nu} = \partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\alpha}^\alpha + \Gamma_{\alpha\beta}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\nu\beta}^\alpha \Gamma_{\alpha\mu}^\beta, \quad (546)$$

where

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\mu} (g_{\beta\mu,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \quad (547)$$

are the Christoffel symbols.

The variation of Eq. (544) is given by

$$\begin{aligned} S_{EH}[g + \varepsilon f] - S_{EH}[g] = & \varepsilon \int d^4x \left(\frac{\partial}{\partial g^{\mu\nu}} \sqrt{-g} g^{\alpha\beta} \right) R_{\alpha\beta} f^{\mu\nu}(x) \\ & + \int d^4x \sqrt{-g} g^{\alpha\beta} \left(R_{\alpha\beta}[g + \varepsilon f] - R_{\alpha\beta}[g] \right) + O(\varepsilon^2). \end{aligned} \quad (548)$$

By calculating in geodesic coordinates one finds that the last term vanishes. Since

$$\frac{\partial}{\partial g^{\mu\nu}} \sqrt{-g} g^{\alpha\beta} = \frac{1}{2\sqrt{-g}} g g_{\mu\nu} g^{\alpha\beta} + \sqrt{-g} \delta_\mu^\alpha \delta_\nu^\beta$$

$$= \sqrt{-g} \left(-\frac{1}{2} g_{\mu\nu} g^{\alpha\beta} + \delta_\mu^\alpha \delta_\nu^\beta \right), \quad (549)$$

we finally obtain

$$S_{EH}[g + \varepsilon f] - S_{EH}[g] = \varepsilon \int d^4x \sqrt{-g} \left(-\frac{1}{2} g_{\alpha\beta} R + R_{\alpha\beta} \right) f^{\alpha\beta}(x) + O(\varepsilon^2). \quad (550)$$

This implies Einstein's field equations in vacuum

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0. \quad (551)$$

For this reason the Lagrangian

$$L_{EH} = -\frac{2}{\kappa^2} \sqrt{-g} R \quad (552)$$

can be taken as starting point of the classical theory.

A glance at Eq. (546) and Eq. (547) shows that the first two terms in Eq. (546) contain second derivatives of the fundamental tensor field $g_{\mu\nu}$. This defect can be removed by splitting off a divergence. We rewrite the first term in Eq. (546) as

$$\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu,\alpha}^\alpha = (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu}^\alpha)_{,\alpha} - \Gamma_{\mu\nu}^\alpha (\sqrt{-g} g^{\mu\nu})_{,\alpha} \quad (553)$$

and calculate the last derivative with the help of

$$g^{\mu\nu}_{,\alpha} = -\Gamma_{\beta\alpha}^\mu g^{\beta\nu} - \Gamma_{\alpha\beta}^\nu g^{\beta\mu}. \quad (554)$$

Proceeding with the second term in the same way we find

$$\sqrt{-g} R = \sqrt{-g} G - (\sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu}^\alpha - \sqrt{-g} g^{\mu\alpha} \Gamma_{\mu\nu}^\nu)_{,\alpha} \quad (555)$$

where

$$G = g^{\mu\nu} \left(\Gamma_{\nu\beta}^\alpha \Gamma_{\mu\alpha}^\beta - \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\beta \right). \quad (556)$$

Since the divergence in Eq. (555) does not matter in the variational principle, we can go on with the Lagrangian

$$L = -\frac{2}{\kappa^2} \sqrt{-g} g^{\mu\nu} \left(\Gamma_{\nu\beta}^\alpha \Gamma_{\mu\alpha}^\beta - \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\beta \right), \quad (557)$$

which contains first derivatives of g only.

For the following it is convenient to remove the square root $\sqrt{-g}$ by introducing the so-called Goldberg variables

$$\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}, \quad \tilde{g}_{\mu\nu} = (-g)^{-1/2} g_{\mu\nu}. \quad (558)$$

Using

$$\begin{aligned} \partial_\rho g &= g \tilde{g}_{\alpha\beta} \tilde{g}^{\alpha\beta}_{,\rho} \\ \partial_\rho g^{\mu\nu} &= (-g)^{-1/2} \left(\tilde{g}^{\mu\nu}_{,\rho} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{g}_{\alpha\beta} \tilde{g}^{\alpha\beta}_{,\rho} \right), \\ \partial_\rho g_{\mu\nu} &= \sqrt{-g} \left(\frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}_{\alpha\beta} \tilde{g}^{\alpha\beta}_{,\rho} - \tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} \tilde{g}^{\alpha\beta}_{,\rho} \right) \end{aligned} \quad (559)$$

in Eq. (547) we obtain

$$\begin{aligned} \Gamma_{\beta\gamma}^\alpha &= \frac{1}{2} \left(\frac{1}{2} \delta_\beta^\alpha \tilde{g}_{\mu\nu} \tilde{g}^{\mu\nu}_{,\gamma} + \frac{1}{2} \delta_\gamma^\alpha \tilde{g}_{\mu\nu} \tilde{g}^{\mu\nu}_{,\beta} - \tilde{g}_{\beta\mu} \tilde{g}^{\alpha\mu}_{,\gamma} \right. \\ &\quad \left. - \tilde{g}_{\gamma\mu} \tilde{g}^{\alpha\mu}_{,\beta} + \tilde{g}^{\alpha\rho} \tilde{g}_{\gamma\mu} \tilde{g}_{\beta\nu} \tilde{g}^{\mu\nu}_{,\rho} - \frac{1}{2} \tilde{g}^{\alpha\rho} \tilde{g}_{\beta\gamma} \tilde{g}_{\mu\nu} \tilde{g}^{\mu\nu}_{,\rho} \right). \end{aligned} \quad (560)$$

This enables us to express the Lagrangian L in Eq. (557) by $\tilde{g}_{\mu\nu}$. It is simple to compute the second term

$$\tilde{g}^{\mu\nu}\Gamma_{\mu\nu}^{\alpha}\Gamma_{\alpha\beta}^{\beta} = -\frac{1}{2}\tilde{g}^{\mu\alpha}{}_{,\mu}\tilde{g}_{\nu\beta}\tilde{g}^{\nu\beta}{}_{,\alpha}. \quad (561)$$

But the first term in Eq. (557) requires the collection of many terms, until one arrives at the simple result

$$\begin{aligned} \tilde{g}^{\mu\nu}\Gamma_{\nu\beta}^{\alpha}\Gamma_{\mu\alpha}^{\beta} = & \frac{1}{4}\left(-2\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu}{}_{,\alpha}\tilde{g}^{\alpha\beta}{}_{,\beta} + 2\tilde{g}_{\alpha\beta}\tilde{g}^{\alpha\mu}{}_{,\nu}\tilde{g}^{\beta\nu}{}_{,\mu} \right. \\ & \left. - \tilde{g}_{\alpha\rho}\tilde{g}_{\beta\sigma}\tilde{g}^{\mu\nu}\tilde{g}^{\rho\beta}{}_{,\mu}\tilde{g}^{\sigma\alpha}{}_{,\nu} + \frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu}{}_{,\alpha}\tilde{g}_{\rho\sigma}\tilde{g}^{\rho\sigma}{}_{,\beta}\right). \end{aligned} \quad (562)$$

Then the total Lagrangian is given by

$$\begin{aligned} L = & \frac{1}{\kappa^2}\left(-\tilde{g}_{\alpha\beta}\tilde{g}^{\alpha\mu}{}_{,\nu}\tilde{g}^{\beta\nu}{}_{,\mu} + \frac{1}{2}\tilde{g}_{\alpha\rho}\tilde{g}_{\beta\sigma}\tilde{g}^{\rho\beta}{}_{,\mu}\tilde{g}^{\alpha\sigma}{}_{,\nu}\tilde{g}^{\mu\nu} \right. \\ & \left. - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu}{}_{,\alpha}\tilde{g}_{\rho\sigma}\tilde{g}^{\rho\sigma}{}_{,\beta}\tilde{g}^{\alpha\beta}\right). \end{aligned} \quad (563)$$

To make contact with quantum field theory on Minkowski space we consider the situation in scattering theory where at large distances in space and time the geometry is flat and given by the Minkowski metric $\eta^{\mu\nu}$. Then we write the metric tensor as a sum

$$\tilde{g}^{\mu\nu}(x) = \eta^{\mu\nu} + \kappa h^{\mu\nu}(x). \quad (564)$$

We do not assume that the new dynamical field $h^{\mu\nu}(x)$ is small in some sense, it only goes to zero at large distances because of the asymptotically flat situation. The indices of $h^{\mu\nu}$ are ordinary Lorentz indices which can be raised and lowered with the Minkowski metric. Then the inverse of Eq. (564) is given by

$$\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} - \kappa h_{\mu\nu}(x) + \kappa^2 h_{\mu\alpha}h^{\alpha}{}_{\nu} - \dots \quad (565)$$

Substituting these expressions into Eq. (563), the Lagrangian L becomes an infinite sum

$$L = \sum_{n=0}^{\infty} \kappa^n L^{(n)}. \quad (566)$$

Here is the proliferation of couplings which is typical for gravity. It can be traced back to the infinite series Eq. (565). The three terms in Eq. (563) give the contributions

$$L_1 = \left(-\eta_{\alpha\beta} + \kappa h_{\alpha\beta} - \kappa^2 h_{\alpha\rho}h^{\rho}{}_{\beta} + \dots\right)h^{\alpha\mu}{}_{,\nu}h^{\beta\nu}{}_{,\mu} \quad (567)$$

$$\begin{aligned} L_2 = & \frac{1}{2}\left(\eta_{\alpha\rho} - \kappa h_{\alpha\rho} + \kappa^2 h_{\alpha\alpha'}h^{\alpha'}{}_{\rho} - \dots\right)\left(\eta_{\beta\sigma} - \kappa h_{\beta\sigma} + \kappa^2 h_{\beta\beta'}h^{\beta'}{}_{\sigma} - \dots\right) \\ & \times (\eta^{\mu\nu} + \kappa h^{\mu\nu})h^{\rho\beta}{}_{,\mu}h^{\alpha\sigma}{}_{,\nu} \end{aligned} \quad (568)$$

$$\begin{aligned} L_3 = & \frac{1}{4}\left(-\eta_{\mu\nu} + \kappa h_{\mu\nu} - \kappa^2 h_{\mu\alpha'}h^{\alpha'}{}_{\nu} - \dots\right)\left(\eta_{\rho\sigma} - \kappa h_{\rho\sigma} + \kappa^2 h_{\rho\beta'}h^{\beta'}{}_{\sigma} - \dots\right) \\ & \times (\eta^{\alpha\beta} + \kappa h^{\alpha\beta})h^{\mu\nu}{}_{,\alpha}h^{\rho\sigma}{}_{,\beta}. \end{aligned} \quad (569)$$

The lowest order

$$L^{(0)} = \frac{1}{2}h^{\alpha\beta}{}_{,\mu}h^{\mu}{}_{\alpha\beta} - h^{\alpha\beta}{}_{,\mu}h^{\mu}{}_{\alpha,\beta} - \frac{1}{4}h_{,\alpha}h^{\alpha}, \quad (570)$$

where $h = h^{\mu}{}_{\mu}$, defines the free theory. Indeed, the corresponding Euler-Lagrange equations reads

$$\square h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\square h - h^{\alpha\mu,\beta}{}_{,\mu} - h^{\beta\mu,\alpha}{}_{,\mu} = 0. \quad (571)$$

Both Eq. (570) and Eq. (571) are invariant under the classical gauge transformation

$$h^{\alpha\beta} \rightarrow \tilde{h}^{\alpha\beta} = h^{\alpha\beta} + f^{\alpha,\beta} + f^{\beta,\alpha} - \eta^{\alpha\beta} f^\mu{}_{,\mu}. \quad (572)$$

The gauge can be specified by the Hilbert condition

$$\tilde{h}^{\alpha\beta}{}_{,\beta} = 0. \quad (573)$$

This can be achieved by choosing the solution of the inhomogeneous wave equation

$$\square f^\alpha = -h^{\alpha\beta}{}_{,\beta}$$

as gauge function in Eq. (572). In the Hilbert gauge the equation of motion Eq. (571) gets simplified

$$\square h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\square h = 0. \quad (574)$$

Taking the trace we conclude

$$\square h = 0, \quad \square h^{\alpha\beta} = 0, \quad (575)$$

so that we precisely arrive at the free tensor field as it was assumed in the QFT.

The first order coupling $O(\kappa)$ in Eqns. (567)-(569) can easily be computed

$$\begin{aligned} L^{(1)} = & -\frac{1}{4}h^{\alpha\beta}h_{,\alpha}h_{,\beta} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}{}_{,\mu}h_{\alpha\beta,\nu} + h_{\alpha\beta}h^{\alpha\mu}{}_{,\nu}h^{\beta\nu}{}_{,\mu} \\ & + \frac{1}{2}h_{\mu\nu}h^{\mu\nu}{}_{,\alpha}h'^{\alpha} - h_{\mu\nu}h^{\alpha\mu}{}_{,\rho}h^{\nu,\rho}_{\alpha}. \end{aligned} \quad (576)$$

The first three terms herein agree precisely with the first three terms in Eq. (542). The last two terms and the forth and fifth terms in Eq. (542) are divergences. This is due to Lorentz contraction of the two derivatives. Indeed, if f_1, f_2, f_3 are massless fields satisfying the wave equation, then the following identity is true

$$2\partial_\alpha f_1 \partial^\alpha f_2 f_3 = \partial^\alpha (\partial_\alpha f_1 f_2 f_3 + f_1 \partial_\alpha f_2 f_3 - f_1 f_2 \partial_\alpha f_3). \quad (577)$$

Since divergence couplings do not change the physics, the coupling Eq. (542) derived from spin-2 quantum gauge theory agrees with general relativity in lowest order. It agrees at higher orders, too (see [6], sect. 5.7). The gauge principle even works in massive gravity [38]. The cohomological nature of gauge invariance was analyzed in [39].

The approach presented above is perturbative in nature and lives on a trivial background. Presently, no fully satisfactory theory of quantum gravity exists, and other ambitious approaches like, e.g., loop quantum gravity aim at a formally background independent description of quantum gravity, and they are expected to give rise to spacetime itself at distances which are large compared to the Planck length. How Einstein's classical geometric view on spacetime is related to such a theory is another story. Here, we content ourselves with the observation that we have found a gauge principle which uses the cohomological formulation of gauge invariance in Eq. (531) etc for the time-ordered products, having the character of a universal principle. Consequently it must be respected in any conventional regularization method.

8 Conclusion

Causality is a fundamental guiding element for the construction of perturbative quantum field theories. Using causality in conjunction with a proper mathematical handling of distribution theory enables one to avoid ultraviolet divergences in perturbative quantum field theory from the start. Whereas standard methods like

dimensional regularization have calculational advantages compared to the causal method, the causal method provides a mathematically well-defined construction scheme of the perturbative S -matrix.

In this review, a condensed introduction and overview of the causal approach to regularization theory has been given, which goes back to a classic paper by Henry Epstein and Vladimir Jurko Glaser [10]. The causal approach was taken up by Michael Dütsch and Günter Scharf in 1985. During the last two decades, several important aspects of the theory have been worked out, which constitute the basis of this review. It should be mentioned that several topics which are not part of this work have been treated in the recent literature, like e.g. interacting fields [40], a complete discussion of perturbative QCD to all orders was worked out [41], and gauge theories like the full standard model (including phenomena like spontaneous symmetry breaking) were studied in [42, 43]. Theories in dimensions other than four were also considered [44] and specific analytic calculations of multi-loop diagrams were carried out [45, 29]. Supersymmetric theories [46] have been investigated, and the causal method was generalized to field theories on curved space-times [47] and studied in the framework of light cone quantum field theory [48].

As mentioned before, there are severe conceptual differences between the causal method and other regularization methods, which make it difficult to compare the different approaches in a reasonable way. Therefore, specific examples have been used in this work in order to demonstrate the differences and connections between the causal and dimensional regularization. On the one hand, dimensional regularization has many attractive features concerning the preservation of gauge invariance and in actual calculations due to its well established methods. On the other hand, the causal method is a strictly mathematical approach without any "intuitive" aspects like continuous spacetime dimensions. Furthermore, the formulation of quantum gauge invariance found during the study of gauge theories in the causal framework has a cogent structure when compared to the standard BRST approach [33, 34]. In this sense, the causal method constitutes an independent framework in its own right with many attractive features. Critical issues like, e.g. axial anomalies can be discussed in a unambiguous manner, and the strong mathematical background of the method permits to apply it to problems on curved spacetime and to quantum gravity, as we have illustrated in the last section.

The mere observation that ultraviolet divergences can be avoided by a proper mathematical construction of Feynman diagrams certainly puts some arguments in the literature concerning the short-range behavior of quantum field theories in connection with ultraviolet divergences into perspective. Several approaches to QFT have been developed so far, and it is obvious that all considerations presented in this review are based on perturbation theory. Even if perturbation theory is well-defined order by order, it is far from being clear that the perturbation expansion can be summed up for physically relevant theories, even Borel summability is most probably not fulfilled due to Landau ghosts and renormalons. Despite these problems, perturbative QFT is a very successful and promising approach, since theoretical predictions of physical quantities made by using renormalized Feynman graph calculations match experimental results with a vertiginous high precision. In all these calculations, one should not think that it is impossible to avoid ill-defined integrals, as the causal approach proves. However, it should be mentioned that formal infinities are admissible if they are treated within a rigorous mathematical framework. Recent developments by Alain Connes and Dirk Kreimer based on Hopf algebras [49] have lead to some profound understanding how to "absorb" ultraviolet divergences in a consistent manner by a redefinition of the parameters defining the QFT. The Hopf algebra approach has also been applied to the causal Epstein-Glaser approach in [50], in order to overcome the separation between the causal method and mainstream QFT.

Richard Feynman in his Nobel lecture remarked: "I think that the renormalization theory is simply a way to sweep the difficulties of the divergencies of electrodynamics under the rug." This problem has been solved by the causal method, at least on a perturbative level.

A Special Distributions in 3+1-Dimensional Spacetime

In this appendix, we give a condensed account of the most important properties of the causal commutators and propagators used in the present review. The distributions used in the causal framework typically differ from the most common definitions found in the literature and by a sign or a normalization factor, since we use the "mathematical" symmetric definition of the (inverse) Fourier transform. Accordingly, the distributions used in the present text are related to the distributions below by the simple redefinitions

$$(2\pi)^2 \hat{D}_{F,ret}^m(k) = -\hat{\Delta}_m^{F,ret}(k), \quad (2\pi)^2 \hat{D}_m^{(\pm)}(k) = -\hat{\Delta}_m^{\pm}(k) \quad (578)$$

in momentum space and by $D_{F,ret}^m(x) = -\Delta_m^{F,ret}(x)$, $D_m^{(\pm)}(x) = -\Delta_m^{\pm}(x)$ in real space, omitting potential mass indices.

The free (non-interacting) neutral scalar quantum field $\varphi(x)$ for particles with a given mass m is given by ($kx = k_\mu x^\mu$, $k^0 = \sqrt{\vec{k}^2 + m^2}$)

$$\varphi(x) = \varphi^-(x) + \varphi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k^0}} [a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{+ikx}], \quad (579)$$

where $\varphi^-(x)$ and $\varphi^+(x)$ refer to the corresponding frequency parts, respectively, whereas the charged field $\varphi_c(x)$ has the form

$$\varphi_c(x) = \varphi_c^-(x) + \varphi_c^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k^0}} [a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{+ikx}]. \quad (580)$$

The commutators of the operator-valued distributions $a(\vec{k})$, $b(\vec{k})$ ("annihilation operators") and $a^\dagger(\vec{k})$, $b^\dagger(\vec{k})$ ("creation operators") are

$$[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k} - \vec{k}'), \quad (581)$$

$$[a(\vec{k}), a(\vec{k}')] = [b(\vec{k}), b(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = [b^\dagger(\vec{k}), b^\dagger(\vec{k}')] = 0 \quad \forall \vec{k}, \vec{k}', \quad (582)$$

and the annihilation operators destroy the unique perturbative vacuum $|0\rangle$ according to $a(\vec{k})|0\rangle = b(\vec{k})|0\rangle = 0 \quad \forall \vec{k}$.

The commutation relations of the scalar fields lead to the so-called positive and negative frequency Jordan-Pauli distributions

$$\Delta_m^\pm(x) = -i[\varphi^\mp(x), \varphi^\pm(0)] = -i\langle 0 | [\varphi^\mp(x), \varphi^\pm(0)] | 0 \rangle, \quad (583)$$

with the distributional Fourier transforms

$$\hat{\Delta}_m^\pm(k) = \int d^4x \Delta_m^\pm(x) e^{ikx} = \mp(2\pi i) \Theta(\pm k^0) \delta(k^2 - m^2). \quad (584)$$

δ is the one-dimensional Dirac distribution depending on $k^2 = k_\mu k^\mu = (k^0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2 = k_0^2 - \vec{k}^2$. The fact that the commutator

$$[\varphi(x), \varphi(0)] = i\Delta_m^+(x) + i\Delta_m^-(x) =: i\Delta_m(x) \quad (585)$$

vanishes for spacelike arguments $x^2 < 0$ due to the requirement of microcausality, leads to the important property that the Jordan-Pauli distribution Δ_m has *causal support*, i.e. it vanishes outside the closed forward and backward light-cone such that

$$\text{supp } \Delta_m(x) \subseteq \bar{V}^- \cup \bar{V}^+, \quad \bar{V}^\pm = \{x \mid x^2 \geq 0, \pm x^0 \geq 0\} \quad (586)$$

in the sense of distributions.

The retarded propagator $\Delta_m^{ret}(x)$ is defined in configuration space by

$$\Delta_m^{ret}(x) = \Theta(x^0)\Delta_m(x), \quad (587)$$

leading to the Fourier transformed expression

$$\hat{\Delta}_m^{ret}(k) = \frac{1}{k^2 - m^2 + ik^0}. \quad (588)$$

The Feynman propagator is given in configuration space by the vacuum expectation value

$$\Delta_m^F(x) = -i\langle 0|T(\varphi_c(x)\varphi_c^\dagger(0))|0\rangle = -i\langle 0|T(\varphi_c^\dagger(x)\varphi_c(0))|0\rangle - i\langle 0|T(\varphi(x)\varphi(0))|0\rangle, \quad (589)$$

the well-known distributional Fourier transform reads

$$\hat{\Delta}_m^F(k) = \frac{1}{k^2 - m^2 + i0}. \quad (590)$$

In the massless case, one has

$$\Delta_0^F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 + i0} = \frac{i}{4\pi^2} \frac{1}{x^2 - i0} = \frac{i}{4\pi^2} P \frac{1}{x^2} - \frac{1}{4\pi} \delta(x^2), \quad (591)$$

where T is the time-ordering operator, P denotes principal value regularization.

It is straightforward to show that the distributions introduced above fulfill the distributional differential equations displayed below. From the wave equation $(\square + m^2)\varphi^{(\pm)}(x) = (\square + m^2)\varphi_c^\pm(x) = 0$ follows

$$(\square + m^2)\Delta_m^\pm(x) = (\square + m^2)\Delta_m(x) = 0. \quad (592)$$

Furthermore, one has

$$(\square + m^2)\Delta_m^F(x) = (\partial_\mu\partial^\mu + m^2)\Delta_m^F(x) = -\delta^{(4)}(x), \quad (593)$$

and

$$(\square + m^2)\Delta_m^{ret}(x) = -\delta^{(4)}(x). \quad (594)$$

The Feynman propagator and the retarded propagator are related via

$$\Delta_m^{ret} = \Delta_m^F + \Delta_m^-. \quad (595)$$

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