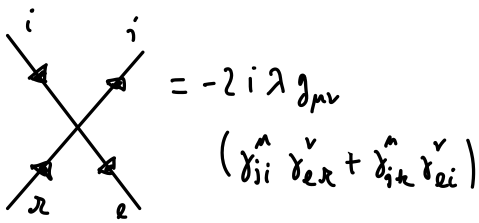
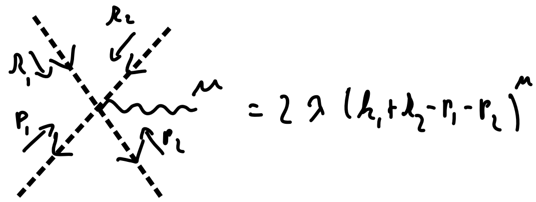
Solutions [and grading guidelines]
for the midterm exam in FYS5120

[General grading guidelines: Stated points are given for arriving at the respective expression in a fully satisfactory way. Whole point subtractions for bad logic, physically wrong statements or expressions that are *physically* wrong (e.g. wrong dimensions, energy not conserved etc.)! 'Obvious' small math errors (e.g. wrong prefactors) only 0.5 pt. When possible, no subtraction for follow-up mistakes. Up to 1 extra point for outstanding explanations or demonstrating special insight (but this cannot result in more points than available for a given problem).]

Problem 1

a) Since the mass dimension of $\mathcal{L}$ must be 4, we have $[\lambda] = -2$ in both cases **[1 pt]**. Assuming all particles to be ingoing, the vertex rules are given by: **[1 pt per correct diagram, including arrow directions, 1 pt per correct rule]**





Important things to note here are: i) a symmetry factor of 2! for each field appearing twice, ii) the replacement of $\partial_\mu \to -ip_\mu$ for ingoing momenta, iii) getting the spinor structure in 1. correct requires adding spinor indices explicitly in the interaction Lagrangian before reading off the vertex and iv) the result in 2. requires to re-shuffle the derivatives first, discarding as usual total divergences, in order to have them act on only one field, respectively.

b) Classical fields evolve according to the classical equations of motion, while quantum fields only do so 'on average' – where the probability to deviate from the classical path is exponentially suppressed by how much the corresponding action deviates from the classical (extremal) action **[1 pt]**. When calculating correlation functions in the path integral formalism, we should thus only perform the integration over the quantum fields, while leaving any given classical field configurations fixed. Vertex rules are based on correlation functions of *quantum* fields – and our formalism to extract them (as used in problem 1a) remains completely unchanged when treating the classical fields as 'coefficients' of the corresponding field operators; for the case of the Yukawa interaction, this results in the vertex stated in the problem (note the missing minus sign in the exam, this is a typo) **[1 pt]**. In the standard model, the Higgs field acquires a non-zero vev, $\phi_{\rm cl} = v$, at small temperatures. Thereby the two-point correlation function for fermions gets an additional constant contribution – i.e. a *mass* term (which would otherwise be forbidden by gauge symmetry) **[1 pt]**.

Problem 2

a) This Lagrangian is invariant under **[1 pt each for the first three; only 0.5 if**

charge is missing]

$$\begin{aligned}
\phi &\to e^{iQ_\phi\alpha}\phi\,, & (1)\\
B_\mu &\to B_\mu + g^{-1}\partial_\mu\alpha & (2)\\
\Psi_L &\to e^{iQ_\psi\alpha}\Psi_L\,, & (3)\\
(\Psi_R &\to \Psi_R)\,, & (4)
\end{aligned}$$

for arbitrary space-time functions $\alpha(x)$. A gauge boson mass term $m_B^2 B_\mu B^\mu$ would not be invariant under (2) **[0.5 pt]**, while a fermion mass term $m\bar\psi\psi = m\bar\psi_L\psi_R + m\bar\psi_R\psi_L$ is forbidden by (3,4) **[0.5 pt]**.

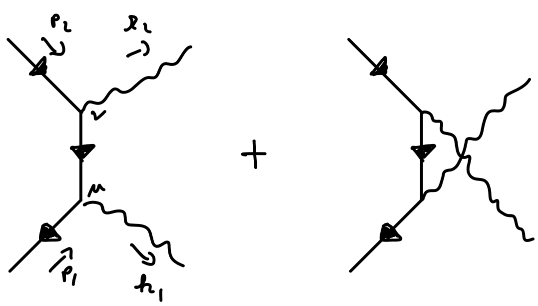
b) The field ϕ has to be expanded around the classical minimum energy solution $\langle\phi\rangle$, which has the consequence that the $U(1)$ symmetry is spontaneously broken, resulting in a massive real scalar and a massless real scalar (the Goldstone boson) **[1 pt]**. Because this symmetry is a *gauge* symmetry, the Goldstone boson is 'eaten' by the gauge field A_μ, which thereby acquires a mass **[1pt]**. This is known as the *Higgs mechanism* **[0.5pt]**.

The photon mass term results from the term $\mathcal{L} \supset |D\phi|^2$. It is most easily derived by assuming that ϕ is *real*-valued – which can always be achieved via a suitable gauge transformation $\alpha(x)$ such that $\exp[iQ_\phi\alpha(x)]\phi$ is real. Then:

$$\begin{aligned}
|D\phi|^2 &= (\partial_\mu\phi + igQ_\phi B_\mu\phi)(\partial^\mu\phi - igQ_\phi B^\mu\phi)\\
&= (\partial_\mu\phi)^2 + g^2Q_\phi^2\phi^2 B_\mu B^\mu\,.
\end{aligned}$$

For $\phi \to \langle\phi\rangle \equiv v$, the second term thus looks like a photon mass term, $\mathcal{L} \supset \frac{1}{2}m_B^2 B_\mu B^\mu$, with $m_B = \sqrt{2}gQ_\phi v$ **[1.5 pts for correct expression and derivation]**.

c) The two diagrams that contribute to this process are **[1 pt]**:



The amplitude is hence given by **[1 pt]**

$$\begin{aligned}
i\mathcal{M} &= \bar v(p_1)\Bigg\{(-igQ_\psi\gamma^\mu P_L)\frac{i(\not k_1 - \not p_1)}{(p_1-k_1)^2}(-igQ_\psi\gamma^\nu P_L) + & (5)\\
&\qquad (-igQ_\psi\gamma^\nu P_L)\frac{i(\not k_2 - \not p_1)}{(p_1-k_2)^2}(-igQ_\psi\gamma^\mu P_L)\Bigg\}u(p_2)\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\\
&= ig^2Q_\psi^2\bar v(p_1)\left\{\frac{\gamma^\mu(\not k_1 - \not p_1)\gamma^\nu P_L}{t} + \frac{\gamma^\nu(\not k_2 - \not p_1)\gamma^\mu P_L}{u}\right\}u(p_2)\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\,, & (6)
\end{aligned}$$

where in the second step we used $P_{R,L}\gamma^\mu = \gamma^\mu P_{L,R}$ **[0.5 pt for full simplification as stated here]**. According to the *Goldstone boson equivalence theorem*, the production of gauge bosons with longitudinal polarizations is identical, in the high-energy limit, to a situation where real Goldstones *would* be emitted **[0.5 pt]**. However, these Goldstone modes must be part of the scalar field in the unbroken theory; and since there does not exist any vertex that couples ψ and ϕ, such a process is not possible **[1 pt]**. (Note that this argument does not involve a fundamental symmetry and hence should not be expected to hold beyond tree-level. In fact, a $\psi\bar{\psi}$ pair can for example annihilate into a $\phi\phi^*$ pair via a gauge boson loop.)

d) The formulation in the problem implies that we need to replace $-igQ_\phi\gamma^\mu P_L\epsilon^*_\mu(k) \to gQ_\phi \not{k} P_L/m_B$ in the amplitude from c), for outgoing gauge boson or scalar π momentum k **[1 pt]**. The amplitude thus becomes **[1 pt]**

$$
\begin{aligned}
i\mathcal{M} &= i\frac{g^2Q_\phi^2}{m_B^2}\bar{v}(p_1)\left\{\frac{\not{k}_1P_L(\not{k}_1-\not{p}_1)\not{k}_2P_L}{(p_1-k_1)^2}+\frac{\not{k}_2P_L(\not{k}_2-\not{p}_1)\not{k}_1P_L}{(p_1-k_2)^2}\right\}u(p_2) \qquad (7)\\
&= -i\frac{1}{2v^2}\bar{v}(p_1)\left\{\frac{\not{k}_1\not{p}_1\not{k}_2P_L}{t}+\frac{\not{k}_2\not{p}_1\not{k}_1P_L}{u}\right\}u(p_2)\,, \qquad (8)
\end{aligned}
$$

where in the second step we used $\not{k}^2 \propto k^2 = 0$. In fact, since *all* momenta are for massless particles, we have $t = -2p_1\cdot k_1$ and $u = -2p_1\cdot k_2$; using the hint, this will simplify expressions like $\not{p}_1\not{k}_1\not{p}_1 = -\not{p}_1\not{p}_1\not{k}_1 + 2\not{p}_1(p_1\cdot k_1) = -t\not{p}_1$ and $\not{p}_1\not{k}_2\not{p}_1 = -u\not{p}_1$ **[1 pt]**. Squaring the matrix element, and averaging over the initial fermion spins, then suddenly becomes doable **[1 pt for any significant simplification of the trace]**:

$$
\begin{aligned}
\overline{|\mathcal{M}|}^2 &= \frac{1}{4\cdot 4v^4}\mathrm{Tr}\left[\not{p}_1\left\{\frac{\not{k}_1\not{p}_1\not{k}_2P_L}{t}+\frac{\not{k}_2\not{p}_1\not{k}_1P_L}{u}\right\}\not{p}_2\left\{\frac{P_R\not{k}_2\not{p}_1\not{k}_1}{t}+\frac{P_R\not{k}_1\not{p}_1\not{k}_2}{u}\right\}\right] \qquad (9)\\
&= \frac{1}{16v^4}\mathrm{Tr}\left[-\not{p}_1\left\{\not{k}_2+\not{k}_1\right\}\not{p}_2P_R\left\{\frac{\not{k}_2\not{p}_1\not{k}_1}{t}+\frac{\not{k}_1\not{p}_1\not{k}_2}{u}\right\}\right] \qquad (10)\\
&= \frac{1}{16v^4}\mathrm{Tr}\left[\left\{\not{k}_2+\not{k}_1\right\}\not{p}_2P_R\left\{\not{k}_2+\not{k}_1\right\}\not{p}_1\right]\,, \qquad (11)
\end{aligned}
$$

where we used the cyclic property of traces in the last step. With momentum conservation, $k_1+k_2 = p_1+p_2$, and using again $(\not{p}_i)^2 = 0$, the trace becomes proportional to $\mathrm{Tr}[\not{p}_1\not{p}_2P_R\not{p}_2\not{p}_1] = \mathrm{Tr}[\not{p}_1\not{p}_2\not{p}_2P_L\not{p}_1] = 0$. In other words, the unpolarized cross section *vanishes* identically (at tree-level) **[1 pt]**.

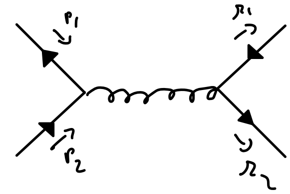
e) For a broken theory, the would-be Goldstone bosons can *always* be extracted by the prescription from problem d), as a consequence of the characteristic *shift symmetry* $\pi(x) \to \pi(x) + const.$ of Goldstone bosons **[1 pt]**. Therefore, the explicit calculation in d) only confirms the general conclusion obtained in c) **[0.5 pt]**. The additional insight from c) is that $m_\psi \neq 0$ would generally result in a non-vanishing $\overline{|\mathcal{M}|}^2$ (adding, e.g. additional terms proportional to m_ψ from the fermion spin sums; the Goldstone modes, on the other hand, must by construction remain massless – so the calculation still remains doable by hand, though it's more tedious) **[0.5 pt]**. In general, it is *not possible* to generate a mass term; this would have to come from a term proportional to $\bar{\psi}\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R$ in the Lagrangian – which picks up a factor of $\exp[\pm iQ_\psi\alpha]$ that cannot be compensated for by either i) further terms involving fermion bilinears (which would result in something different from a mass term) ii) any integer power of scalar fields (only adding multiple factors of of $\exp[\pm iQ_\phi\alpha]$) **[1 pt]**. If, however, one picks the specific charge $Q_\phi = Q_\psi$ for the scalar field, the situation changes and the dim-4 Yukawa coupling $\mathcal{L} \supset -\lambda\phi\bar{\psi}\psi$ becomes possible – very similar, in fact, to the SM case. When ϕ acquires a vev, this will generate a fermion mass term $m_\psi = \lambda v$ **[1 pt]** – and at the same time add a Goldstone-fermion coupling that allows the tree-level production of longitudinal gauge bosons, cf. problems c), d) **[0.5 pt]**. The rate for this production will be directly proportional to $(m_\psi/m_B)^4 \propto (\lambda/g)^4$, and hence independent of v **[0.5 pt]**.

Problem 3

a) The gauge group of the standard model is given by $G = U(1)_Y \times SU(2)_L \times SU(3)_c$ **[0.5 pt]**. As gauge bosons transform in the adjoint representation, which has $N^2 - 1$ generators for $SU(N)$ **[1 pt]**, this implies that there are 8 gauge bosons for $SU(3)$ (the gluons), 3 electroweak gauge bosons for $SU(2)$ and the weak hypercharge boson **[0.5 pt]**. Gauge symmetry forbids mass terms, so each of these contributes with 2 physical d.o.f., i.e. 24 in total **[0.5 pt]**. The Higgs field is by construction only charged under $U(1)_Y \times SU(2)$, and must transform in the same representation as the fermions. This constrains it to be a complex doublet, thus adding further 4 (real) degrees of freedom. **[0.5 pt]**
When the Higgs field acquires a vacuum expectation value, it *breaks the original symmetry* as $U(1)_Y \times SU(2)_L \to U(1)_{\text{em}}$ **[0.5pt]**. Three of the original gauge bosons thus acquire a mass by 'eating' the corresponding components of the Higgs doublet (those are the observed $W^\pm$ and Z bosons, with 3 real degrees of freedom each) **[0.5 pt]**. The remaining massless vector boson is the photon, and the remaining d.o.f. of the Higgs doublet is the physical Higgs field; the gluons are unaffected and remain massless **[0.5 pt]**. Both before and after SSB, the total number of bosonic, real d.o.f. in the SM is thus $1 \times 2 + 3 \times 2 + 8 \times 2 + 4 = 1 \times 2 + 3 \times 3 + 8 \times 2 + 1 = 28$. **[0.5 pt]**

b) The *fermion* fields transform in the fundamental representation of the respective gauge group **[0.5 pt]**, i.e. doublets under $SU(2)$ and triplets under $SU(3)$. Both left-handed quarks and left-handed leptons belong to $SU(2)$ doublets **[1 pt]**. The left-handed quarks additionally belong to an $SU(3)$ triplet, just as the right-handed quark fields do **[1 pt]**. A right-handed neutrino is strictly speaking not part of the SM – but would be uncharged under all of the SM gauge interactions **[0.5 pt]**.

c) In the *QCD case*, there is only one contributing diagram, with an s-channel gluon (see the figure to the right) **[1 pt]**. The QCD amplitude (in Feynman gauge) reads **[1 pt]**



$$
\begin{aligned}
i\mathcal{M}_{QCD} &= \bar{v}(p_1)(-ig_s)\gamma^\mu t^a u(p_2)\frac{-ig_{\mu\nu}\delta^{ab}}{(p_1+p_2)^2}\bar{u}(k_2)(-ig_s)\gamma^\nu t^b v(k_1) \qquad (12)\\
&= i\frac{g_s^2}{s}[v(p_1)_i\gamma^\mu u(p_2)_j][\bar{u}(k_2)_k\gamma_\mu v(k_1)_l](t^a)_{ij}(t^a)_{kl}\,, \qquad (13)
\end{aligned}
$$

where the t^a are $SU(N)$ generators (i.e. the Gell-Mann matrices for the $N=3$ case of QCD) and subscripts on the spinors indicate color (not spinor) indices. Averaging over the 2 spin and $N=3$ color states of each of the incoming quarks, and summing over the final state spin and color states gives **[1 pt]**

$$
\begin{aligned}
\overline{|\mathcal{M}_{QCD}|}^2 &= \frac{1}{2^2N^2}\frac{g_s^4}{s^2}\mathrm{Tr}[(\not{p}_1-m_u)\gamma^\mu(\not{p}_2+m_u)\gamma^\nu]\times\mathrm{Tr}[(\not{k}_2+m_d)\gamma_\mu(\not{k}_1-m_u)\gamma_\nu]\\
&\quad\times(t^a)_{ij}(t^b)^*_{ij}(t^a)_{kl}(t^b)^*_{kl}\,. \qquad (14)
\end{aligned}
$$

For the unpolarized *QED amplitude* squared, we only need to replace the virtual gluon with a photon propagator **[1 pt]**, resulting in **[1 pt]**

$$
\overline{|\mathcal{M}_{QED}|}^2 = \frac{N^2}{N^2}\frac{1}{2^2}\frac{Q_u^2Q_d^2e^4}{s^2}\mathrm{Tr}[(\not{p}_1-m_u)\gamma^\mu(\not{p}_2+m_u)\gamma^\nu]\times\mathrm{Tr}[(\not{k}_2+m_d)\gamma_\mu(\not{k}_1-m_u)\gamma_\nu]\,, \qquad (15)
$$

where $Q_u=2/3$ is the electric charge of the up quark and $Q_d=-1/3$ that of the down quark. Note the additional color factor of N^2, coming from the fact that there are three color options for $\bar{q}q$ in both initial and final states (which then is averaged away if we assume that nothing is known about the initial state colors).[1] All kinematic factors are thus identical, and the ratio of the cross sections is hence given by **[2 pt for correct final result]**

$$
\begin{aligned}
\frac{\sigma_{QCD}}{\sigma_{QED}} &= \frac{1}{Q_u^2Q_d^2}\frac{\alpha_s^2}{\alpha_{\mathrm{em}}^2}\frac{1}{N^2}\times(t^a)_{ij}(t^b)^*_{ij}\underbrace{(t^a)_{kl}(t^b)^*_{kl}}_{=(t^a)_{ij}(t^b)_{ji}=\mathrm{Tr}[t^at^b]=C(r)\delta^{ab}} \qquad (16)\\
&= \frac{81}{4}\frac{\alpha_s^2}{\alpha_{\mathrm{em}}^2}\frac{C(r)^2}{N^2}d(G) \qquad (17)\\
&= \frac{81}{4}\frac{\alpha_s^2}{\alpha_{\mathrm{em}}^2}\frac{3^2-1}{4\times 3^2} = \frac{9}{2}\frac{\alpha_s^2}{\alpha_{\mathrm{em}}^2}\,. \qquad (18)
\end{aligned}
$$

[1] An alternative *definition* of unknown initial color states would be that the antiquark always has the anti-color of the quark color. In this case the averaging would be implemented as $N^{-2}\to N^{-1}$ for *both* cross sections, leaving the ratio unchanged.