Torsten Bringmann (torsten.bringmann@fys.uio.no)
http://www.uio.no/studier/emner/matnat/fys/FYS5120/v23/

Lecture spring 2023:

Quantum field theory II

Compulsory problem set

⇝ Deadline for handing in answers is **Monday, 15 Mai 2023**. You must pass this assignment in order to proceed to the final (oral) exam.

For ease of correcting, please try to follow Schwarz' conventions for counter terms, self-energies etc. You don't have to re-derive (intermediate) results quoted in the lecture or previous exercises – so, in order to safe time, make sure to not re-invent the wheel!

Problem I

Consider a theory with two Dirac fermions, ψ_a and ψ_b, that have identical charge under a $U(1)$ interaction with strength g:

$$\mathcal{L}_\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}_a \left(i\not{D} - m_a\right)\psi_a + \overline{\psi}_b \left(i\not{D} - m_b\right)\psi_b$$

a) Which are the 7 (5) counter terms of this theory? State them in the $\overline{\text{MS}}$ scheme, to leading order in $\alpha \equiv g^2/(4\pi)$!

b) Express the pole mass of ψ_a and ψ_b in terms of their $\overline{\text{MS}}$ masses, again at $\mathcal{O}(\alpha)$. Does the theory generate a mass mixing, at higher order in α, i.e. a non-vanishing two-point correlation function $\langle\Omega|\psi_b\overline{\psi}_a|\Omega\rangle$?

c) Why is the theory described by $\mathcal{L}_\psi$ renormalizable to all orders in α ? Use this fact to argue that the correlation function for three gauge bosons, $\langle\Omega|A^\mu A^\nu A^\rho|\Omega\rangle$, must vanish identically – to all orders in perturbation theory! Does a similar argument also work for the case of five gauge bosons, $\langle\Omega|A^{\mu_1}...A^{\mu_5}|\Omega\rangle$?

Problem II

Now add a real massless scalar field ϕ to the theory in problem I, with an off-diagonal Yukawa coupling y to the fermions:

$$\mathcal{L} = \mathcal{L}_\psi + \frac{1}{2}(\partial_\mu\phi)^2 + y\phi\left(\overline{\psi}_a\psi_b + \overline{\psi}_b\psi_a\right) .$$

a) Which *additional* dim ≤ 4 operators does the theory generate at loop level (beyond those stated in $\mathcal{L}$)? Are the coefficients of these operators calculable? If yes: explain how; if not: explain how to treat these operators instead.

b) Based on your insight from a), list all counter terms in this theory and calculate them in the $\overline{\text{MS}}$ scheme (again at leading order)! In the interest of time, you *do not* have to calculate δ_y and the counterterm for the interaction that we study in more detail in problem III.

c) Which dim ≤ 4 operators involving any of the four fields in our theory are in principle possible – by Lorentz invariance – but *not* generated radiatively? For each of them, state the symmetry that forbids them. What about dim > 4 operators in this theory – are they always calculable (and why)?

Problem III

One of the additional operators that you should have identified in IIa) is a quartic scalar self-interaction, $\frac{\lambda}{4!}\phi^4$. In other words, λ must be *measured* at a given energy scale μ_0. From that, however, we can *predict* the effective value of λ at any other energy scale μ.

a) For the rest of this problem, we will assume $m_a = m_b = 0$ for simplicity. Why is this a technically natural choice?

b) How many diagrams contribute to λ that contain a fermion loop? Calculate the divergent part of *one* of them explicitly. From the result, compute the sum over *all* diagrams by using symmetry arguments. Do the same for scalar loops.

c) Compute the counterterm δ_λ in the $\overline{\text{MS}}$ scheme. Calculate the μ-dependence of the $\phi\phi \to \phi\phi$ amplitude, where μ is the renormalization scale in dimensional regularization. Denote this amplitude by Λ, and express it in terms of the same amplitude Λ' that would follow from a different renormalization condition $\mu \to \mu'$. Interpret the result in terms of the introductory claim (about being able to predict the strength of the 4-point coupling).

d) Another possible renormalization condition is to assign to Λ the value of the 4-point coupling one would measure in the extreme low-energy limit where $p_i^\mu = (m_\phi, \mathbf{0})$ for all four external momenta p_i. Compute the effective coupling Λ in this scheme! Can you map this to the $\overline{\text{MS}}$ scheme, i.e. can you identify a value of μ_0 that implements this renormalization condition? How do you interpret the result?