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`http://www.uio.no/studier/emner/matnat/fys/FYS5120/v24/`

Lecture spring 2024:

Quantum field theory II

Here I provide final (and in some rare cases intermediate) *results* **for the problem sets. The sole purpose is to give you the opportunity of cross-checking your own calculations – not to guide you through the details of these calculations. Apart from time constraints, the main reason is that you really have to perform these kinds of calculations on your own in order to learn QFT (and to be prepared for the exam). Also, please remember that we have ample of opportunities to discuss technical aspects about the derivations in the exercise / Q&A sessions as well as on AstroForum – so do make use of this!**

Problem 1

a) With the notation in P&S, you need to use $\Delta\phi = i\phi$. This gives $j^\mu \propto \phi\partial^\mu\phi^* - \phi^*\partial\phi$, from which you get $Q \propto \int d^3x\, \Im(\dot{\phi}^*\phi)$ by integrating over j^0.

b) The Hamiltonian is given by

$$\int d^3x\, (\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi) = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}) + c\,,$$

where c is an infinite constant that does not affect the equations of motion.

c) For the charge, you should find $Q \propto \int \frac{d^3p}{(2\pi)^3}(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}})$. Particles and anti-particles thus have opposite charge.

Problem 2

$$\int d^N p\, e^{-\frac{1}{2}p_m B_{mn} p_n + J_n p_n} = \sqrt{\frac{(2\pi)^N}{\det B}} \exp\left[\frac{1}{2} J_m \left(B^{-1}\right)_{mn} J_n\right] \tag{1}$$

$$\to \int \mathcal{D}\phi\, \exp\left[-i\int d^4x \left(\frac{1}{2}\phi D_x \phi - J\phi\right)\right] \propto \exp\left[-\frac{1}{2}\int d^4x \int d^4y\, J(x) G(x-y) J(y)\right], \tag{2}$$

where D_x is some differential operator and G its Green function. Grassmann version:

$$\int d(\theta^*\theta)^N\, e^{-\theta^*_m B_{mn}\theta_n + \eta^*_n\theta_n + \eta_n\theta^*_n} = (\det B) \exp\left[\theta^*_m \left(B^{-1}\right)_{mn} \theta_n\right] \tag{3}$$

$$\to \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp\left[-i\int d^4x (\bar{\psi} D_x \psi - \bar{\eta}\psi - \bar{\psi}\eta)\right] \propto \exp\left[-\int d^4x \int d^4y\, \bar{\eta}(x) G(x-y) \eta(y)\right], \tag{4}$$

Problem 3

a, b) These are of 'show that' type to fill in small gaps in the lecture...

c)
$$Z_0[J] = \int \mathcal{D}\phi \exp\left[i \int d^4k \left\{-\frac{1}{2}\phi(k)(-k^2+m^2-i\epsilon)\phi(-k) + J(k)\phi(-k)\right\}\right] \tag{5}$$
$$\propto \exp\left[-\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J(-k)G(k)J(k)\right] \tag{6}$$

Therefore,
$$\langle 0|T\{\hat{\phi}(-k)\hat{\phi}(p)\}|0\rangle = \frac{(-i)^2}{Z_0[0]}\frac{\delta}{\delta J(k)}\frac{\delta}{\delta J(-p)} Z_0[J]_{J=0} = G(k)\,\delta^{(4)}(p-k)\,.$$

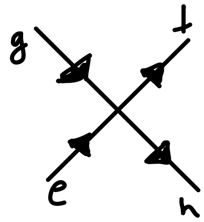
Problem 4

(This is also of 'show that' type. We went through this in quite some detail in the first exercise / Q&A session)

Problem 5

a) "dim 6":

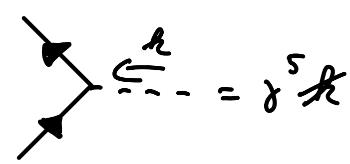
$$\mathcal{L} \supset \lambda g_{\mu\nu} \bar{\psi}_a \gamma^\mu_{ab} \psi_b \bar{\psi}_c \gamma^\nu_{cd} \psi_d$$



$$= 2i\lambda g_{\mu\nu}\{\gamma^\mu_{fe}\gamma^\nu_{hg} + \gamma^\mu_{fg}\gamma^\nu_{he}\}$$

b) "dim 5":

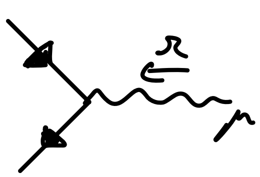
$$\mathcal{L} \supset \lambda(\partial_\mu \phi)\bar{\psi}\gamma^5\gamma^\mu\psi$$



$$= \gamma^5 \not{k}$$

c) "dim 5"

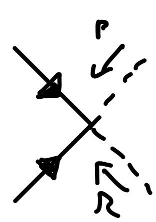
$$\mathcal{L} \supset \lambda F^{\mu\nu}\bar{\psi}\sigma_{\mu\nu}\psi = i\lambda(\partial_\mu A_\nu)\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi$$



$$= 2i\lambda(\not{k}\gamma^\mu - k^\mu)$$

d) "dim 6"

$$\mathcal{L} \supset -2\lambda\phi(\partial_\mu\phi)\bar{\psi}\gamma^\mu\psi$$



$$= -2\lambda(\not{k}+\not{p})$$

Problem 6

a) The Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^* - V(|\phi|)$, with $D_\mu = \partial_\mu - iQ_\phi g A_\mu$, is invariant under

$$\phi \to (1 + iQ_\phi\alpha)\phi \tag{7}$$

$$A_\mu \to A_\mu + \frac{1}{Q_\phi g}\partial_\mu\alpha\,, \tag{8}$$

for arbitrary scalar charge Q_ϕ, coupling strength g and real functions $\alpha(x), V(|\phi|)$.

b) We need to add the term $\mathcal{L} \supset \bar{\psi}(i\not{D})P_R\psi$, indicating the possibility of a different charge by using $D_\mu = \partial_\mu - iQ_\psi g A_\mu$. This Lagrangian is also invariant under the transformations in a), with

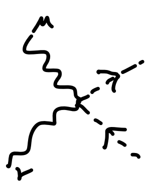
$$P_R\psi \to (1 + iQ_\psi\alpha)P_R\psi \tag{9}$$

$$P_L\psi \to P_L\psi\,. \tag{10}$$

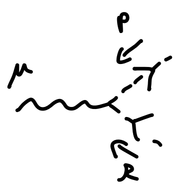
A mass term $m_\psi\bar{\psi}\psi = m_\psi\left[\overline{(P_R\psi)}P_L\psi + \overline{(P_L\psi)}P_R\psi\right]$ would not be invariant under this transformation. Same for $m_A A_\mu A^\mu$.

c) Tuning the charges to $Q_\phi = Q_\psi$ additionally allows the Yukawa-like interaction $\mathcal{L} \subset -y\left(\phi\bar{\psi}P_L\psi + \phi^*\bar{\psi}P_R\psi\right)$. The interaction terms in the full theory, on top of those deriving from $V(|\phi|)$, are then given by

$$|D_\mu\phi|^2 \supset iQgA^\mu(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + Q^2g^2A_\mu^2|\phi|^2$$

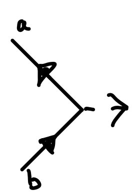
$\rightsquigarrow$ 

$= 2iQ^2g^2 g_{\mu\nu}$

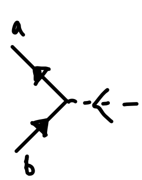


$= iQg(k-p)^\mu$

$$\mathcal{L} \supset -y\phi\bar{\psi}P_L\psi + h.c.$$

$\rightsquigarrow$ 

$= -iy P_L^{ba}$

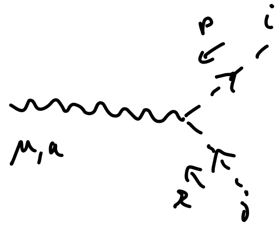


$= -iy P_R^{ab}$

Problem 7

$$D_\mu = \partial_\mu - i g A_\mu^a t^a \Rightarrow (D_\mu \phi)^2 \supset i g A_\mu^a \left[\phi^\dagger \overset{(+)}{t^a} \partial_\mu \phi - (\partial_\mu \phi^\dagger) t^a \phi\right] + g^2 \left|A_\mu^a t^a \phi\right|^2$$

$\rightsquigarrow$



$= i g (k-p)^\mu t^a_{ij}$

$= 2 i g^2 g_{\mu\nu} (t^a t^b)_{ij}$

Problem 8

Finite gauge transformations:

$$\psi \to V(x)\psi \equiv \exp\left(i\alpha^a t^a\right)\psi$$

$$A_\mu^a t^a \to V\left(A_\mu^a t^a + \frac{i}{g}\partial_\mu\right)V^\dagger$$

(The one for the gauge field cannot be brought into a more compact/explicit form).

Problem 9

a) This is all 'show that'. (A projector is anything that satisfies $P^2 = P$, so here $P_{\mu\nu}P^\nu{}_\rho = P_{\mu\rho}$).

b) With $\epsilon^0_\mu = p_\mu/m$ the spin sum follows almost trivially from a). The physical d.o.f. of a massive vector boson correspond to the three possible spin-directions in its rest frame (i.e. $2S+1=3$), while the time-like component is not a dynamical d.o.f. This must be reflected both in possible external states ($\rightsquigarrow$ spin/polarization sums) and internal/ propagating states (virtual states are typically off-shell; but they do not add additional d.o.f. – i.e. real space-time functions – to the theory).

c) The gauge that implements both $k\cdot\epsilon$ and $\mathbf{k}\cdot\epsilon$ is the combined Lorentz and Coulomb gauge. In QED the Ward identity guarantees that any amplitude $\mathcal{M}_\mu\epsilon^\mu(k)$ vanishes when replacing $\epsilon^\mu(k) \to k^\mu$; in particular, the time-like and longitudinal polarizations add exactly equal and opposite to the amplitude. In other words, while it may *look like* we are summing over *all* polarizations in Feynman gauge, we actually still, effectively, only sum over the transverse d.o.f. –

the physical number of d.o.f. is invariant and does not depend on the gauge (or whether a particle is off-shell). For the same reason, interpreting the propagator in Landau gauge in terms of the 3 d.o.f. of a massive vector boson is even more misleading.
In the non-Abelian case, the situation is more complicated as the Ward identity no longer holds. In the Feynman gauge, e.g., this implies that it is no longer only the transverse d.o.f. that contribute to the amplitude – and we need ghosts to subtract the contributions from the unphysical (longitudinal and time-like) states.

a) This result is most easily derived by an analysis of the Lorentz structure. I.e. write it as a sum of all terms that could potentially contribute – namely $g_{\mu\nu}, k^\mu k^\nu, n^\mu n^\nu, n^{(\mu}p^{\nu)}$ – and determine the coefficients by contracting this expression with k^μ or p^μ. It's also worth convincing yourself that the result indeed reproduces that from c) for $n^\mu = (1, \mathbf{0})$...

Problem 10

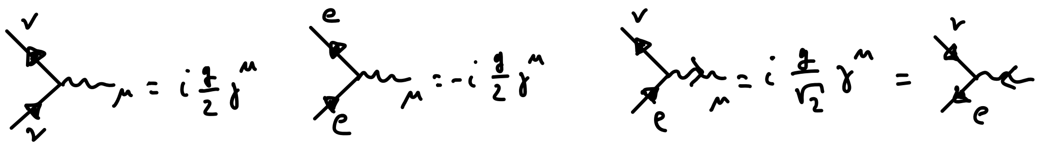
a)
$$\not{D} = \not{\partial} - ig\frac{\sigma^i}{2}\not{A}^i = \begin{pmatrix} \not{\partial} - \frac{ig}{2}\not{A}^3 & -\frac{ig}{\sqrt{2}}\not{W}^+ \\ -\frac{ig}{\sqrt{2}}\not{W}^- & \not{\partial} + \frac{ig}{2}\not{A}^3 \end{pmatrix} \tag{11}$$

b)
$$\begin{aligned} -\frac{1}{4}(F^a_{\mu\nu})^2 = {} & \text{(quadratic terms)} + \\ & + ig\Big\{ A_3^\mu \left(W_+^\nu \partial_\mu W_{-\nu} - W_-^\nu \partial_\mu W_{+\nu}\right) + W_-^\mu \left(A_3^\nu \partial_\mu W_{+\nu} - W_+^\nu \partial_\mu A_{3\nu}\right) \\ & \qquad + W_+^\mu \left(W_-^\nu \partial_\mu A_{3\nu} - A_3^\nu \partial_\mu W_{-\nu}\right) \Big\} \qquad (12) \\ & + g^2\Big\{ \frac{1}{2}W_+^2 W_-^2 - \frac{1}{2}(W^+ \cdot W_-)^2 \\ & \qquad - A_3^2 (W^+ \cdot W_-) + (A_3 \cdot W_+)(A_3 \cdot W_-) \Big\} \qquad (13) \end{aligned}$$

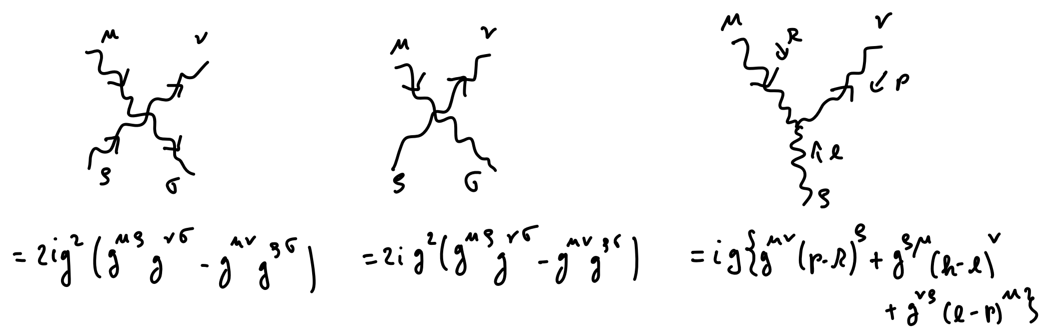
c) Using the covariant derivative from a) the interaction terms from the fermionic part of the Lagrangian read

$$i\overline{\psi}\not{D}\psi \supset \frac{g}{2}\overline{\nu}\not{A}^3\nu + \frac{g}{\sqrt{2}}\overline{\nu}\not{W}^+ e + \frac{g}{\sqrt{2}}\overline{e}\not{W}^-\nu - \frac{g}{2}\overline{e}\not{A}^3 e \tag{14}$$

This leads to the following four Feynman rules (where the arrow of the W lines is chosen such that the charge flow corresponds to that of the electron – i.e. the arrow is defined to point along the momentum direction for a W^-, not a W^+):



The interaction terms from b), on the other hand, correspond to these three vertex rules:



Problem 11

a) You should find $\overline{|\mathcal{M}|}^2 = 2Q^4 e^4 u/t$, where Q is the electric charge of the quark, and hence $d\sigma/dt = 2\pi\alpha_{\rm em}^2 u/(ts^2)$.

b) The additional diagram is an s-channel gauge boson, with a three-boson vertex. Still, for $t \to 0$ only the t-channel diagram diagrams contributes. Averaging over the N possible internal states of the incoming fermions (squared) and summing over the color indices – which gives a result proportional to the quadratic Casimir $C_2 = (N^2-1)/(2N)$ of $SU(N)$ – you should recover the result from a) with an additional factor of $(N^2-1)^2/(4N^3)$, and $Q^4\alpha_{\rm em}^2$ replaced by $\alpha = g^2/(4\pi)$. For QCD ($N = 3$), this gives the result stated in the problem.

c) The mistake due to wrongly assuming that the Ward identity holds can be compensated for by subtracting the cross section that one *would* get for real ghost final states (if ghosts where physical particles). Since ghosts only couple to gauge bosons, the only contribution can come from the s-channel diagram. The general expression for the amplitude of such a diagram is straightforward to derive (see also the lecture); squaring this in the high-energy limit (i.e. neglecting quark masses) and averaging/summing over color d.o.f. leads to an expression that is proportional to the *index* of $SU(N)$ in the fundamental representation (from the quarks) times the index in the adjoint representation (from the ghosts). In total, you should find $d\sigma/dt = \frac{\pi}{4}\alpha^2 \frac{N^2-1}{N}\frac{ut}{s^4}$.

Problem 12

a) The Lagrangian of this theory is described by $\mathcal{L} = |D_\mu\phi|^2 - \frac{1}{4}(F_{\mu\nu})^2$. The contributions from t- and s-channel diagrams then add up to $|\mathcal{M}|^2 = 4g^4(1 + t/s + s/t)^2$, i.e. $d\sigma/dt = 4\pi\alpha^2(1 + t/s + s/t)^2/s^2$.

b) Summing/averaging over the N components turns out to not affect the individual diagrams (squared), only the interference term. The result can be written as

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{s^2}\left[4\left(1 + \frac{t}{s} + \frac{s}{t}\right)^2 - 2\frac{N-1}{N}\left(1 + \frac{2t}{s}\right)\left(1 + \frac{2s}{t}\right)\right]. \tag{15}$$

c) Also in this case, there is a common factor in front of the individual diagrams squared, and another (though very similar) for the interference terms. In simplified form:

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{s^2}\frac{N^2}{N^2-1}\left[4\left(1 + \frac{t}{s} + \frac{s}{t}\right)^2 - \left(1 + \frac{2t}{s}\right)\left(1 + \frac{2s}{t}\right)\right]. \tag{16}$$

Problem 13 This is 'show that'... but the idea is to choose a specific vacuum state such as $\phi = (0, ..., 0, v)$ and check which sub(!)-transformation of the original transformation leaves this state invariant. Then introduce $\phi_N(x) \equiv v + \sigma(x)$ and $\phi_i(x) \equiv \pi_i(x)$ for $1 \leq i \leq n - 1$, and check the terms quadratic in π_i.

Problem 14 See 28.2 in the book by Schwartz... some additional comments:

a) The statement that Q 'generates' the symmetry derives from how a general continuous transformation $V = \exp[i\alpha X]$, with generator X, acts on an *operator* $\mathcal{O}$, namely by

$$\mathcal{O} \to V\mathcal{O}V^{-1} = \mathcal{O} + i\alpha[X, \mathcal{O}] + ... \tag{17}$$

This is to be compared with how the field operator transforms, in the form used in the derivation of the Noether current:

$$\phi_m \to \phi_m + \alpha(\delta\phi_m/\delta\alpha)\,. \tag{18}$$

Given the expression for the commutator $[Q, \phi_m]$, these two equations become identical for $X = Q$ (and $\mathcal{O} = \phi_m$). Thus, a symmetric state is unaffected by Q, i.e. $Q|\psi\rangle = 0$ or, equivalently, $\exp[i\alpha Q]|\psi\rangle = |\psi\rangle$ – while a state without symmetry isn't, $Q|\psi\rangle \neq 0$.

c) $|\pi(\mathbf{0})\rangle \propto Q$ *annihilates* the false (but not true) vacuum. Hence, the result is not a state in the Hilbert space and hence we cannot learn anything about its energy.

Problem 15

a) ψ and ϕ are doublets, and there are three gauge fields. The covariant derivatives are *not* the same:

$$
\begin{aligned}
D_\mu \Psi_L &= \partial_\mu \Psi_L && (19)\\
D_\mu\{\Psi_R, \Phi\} &= \left(\partial_\mu - igA_\mu^a \frac{\sigma^a}{2}\right)\{\Psi_R, \Phi\} && (20)\\
(D_\mu A_\nu)^a &= \partial_\mu A_\nu^a + g\epsilon^{abc} A_\mu^b A_\nu^c && (21)
\end{aligned}
$$

b) Denoting the multiplet components as $\Psi = (\psi_1, \psi_2)$, the missing terms are

$$
\mathcal{L}_{\Phi,\Psi} \supset -\lambda_1 \bar{\Psi}_R \tilde{\Phi} \psi_{1L} - \lambda_2 \bar{\Psi}_R \Phi \psi_{2L} + h.c \tag{22}
$$

In unitary gauge, we have $\Phi(x) = \frac{1}{\sqrt{2}}(0, v + \phi(x))$, giving rise to mass terms

$$
-\frac{\lambda_1 v}{\sqrt{2}} \underbrace{\left(\bar{\psi}_{1L}\psi_{1R} + \bar{\psi}_{1R}\psi_{1L}\right)}_{\bar{\psi}_1\psi_1} - \frac{\lambda_2 v}{\sqrt{2}} \underbrace{\left(\bar{\psi}_{2L}\psi_{2R} + \bar{\psi}_{2R}\psi_{2L}\right)}_{\bar{\psi}_2\psi_2} = -m_1 \bar{\psi}_1\psi_1 - m_2 \bar{\psi}_2 \psi_2\,.
$$

From the $|D_\mu\phi|^2$ term, all gauge boson masses are $m_A = gv/2$ – which is different from the SM (apart from the fact that the gauge bosons couple to right-, rather than left-handed fields). The reason is that here the symmetry is fully broken, while in the SM there is a residual $U(1)_{\mathrm{em}}$ symmetry after breaking the original $U(1)_Y \times SU(2)_L$ symmetry.

c) These interactions are again most easily obtained in unitary gauge, and contained in the term $|D_\mu\phi|^2 \supset \frac{g^2}{8}(v+\phi)^2 A_\mu^a A^{a\,\mu}$. Thus, the 3-point ($AA\phi$) vertex rule is $\frac{i}{2} g^2 v g_{\mu\nu}\delta^{ab}$, and the 4-point ($AA\phi\phi$) vertex rule is $\frac{i}{2} g^2 g_{\mu\nu}\delta^{ab}$. Note how this could have been directly obtained from the result for the gauge boson mass term, with $v \to v + \phi$. Also note the necessary appearance of 'δ^{ab}': the vacuum after SSB is neutral in this model, i.e. not charged under (a subgroup of) $SU(2)$; ϕ must have the same charge as v, so the color index of the two gauge bosons cannot change in these interactions.