Torsten Bringmann (torsten.bringmann@fys.uio.no)
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Lecture spring 2023:

Quantum field theory II

Problem sheet 3

Problem 9

This problem serves to build some intuition about vector boson polarizations, and how their outer product is related to the denominator appearing in propagators. We use a convention with four linearly independent 'polarization vectors' $\epsilon^{(\lambda)}$ that satisfy the orthonormality relation $\epsilon^{(\lambda)*} \cdot \epsilon^{(\lambda')} = g^{\mu\nu} \epsilon_\mu^{(\lambda)*} \epsilon_\nu^{(\lambda')} = g^{\lambda\lambda'}$.

a) The simplest choice of polarization vectors is $\epsilon_\mu^{(\lambda)} = \delta_\mu^\lambda$. Show that this satisfies the above relation, and also that this results in the completeness relation

$$|\epsilon\rangle\langle\epsilon|_{\mu\nu} \equiv g_{\lambda\lambda'} \epsilon_\mu^{(\lambda)*} \epsilon_\nu^{(\lambda')} = g_{\mu\nu} \,. \tag{1}$$

Demonstrate that these relations are frame-independent, i.e. invariant under $\epsilon_\mu^{(\lambda)} \to \Lambda_\mu{}^\nu \epsilon_\nu^{(\lambda)}$, and that $|\epsilon\rangle\langle\epsilon|$ mathematically speaking is indeed a projector (as indicated by the notation).

b) Now consider a time-like 4-momentum that takes the form $p^\mu = (m, \mathbf{0})$ in the frame where $\epsilon_\mu^{(\lambda)} = \delta_\mu^\lambda$. In a general frame, where $p^\mu = (E, \mathbf{p})$, what is the form of the time-like polarization vector, $\epsilon_\mu^0 \equiv \epsilon_\mu^{(0)}$? Show that

$$|\epsilon\rangle\langle\epsilon|_{\mu\nu}^{L,T} \equiv g_{ii'} \epsilon_\mu^{(i)*} \epsilon_\nu^{(i')} = - \sum_{p\cdot\epsilon=0} \epsilon_\mu^* \epsilon_\nu = g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \tag{2}$$

and that this is again a projector, this time on all polarization states *but* the time-like one. Why do you expect the propagator for a *massive* vector boson – which we will derive later in the lecture – to be proportional to this?

c) For a massless particle, on the other hand, the momentum is light-like: $k^\mu = (\omega, \mathbf{k})$ with $|\mathbf{k}| = \omega$. We can then still consider a complete set of polarization vectors, just not in the rest frame. For example a time-like $\epsilon^0 \propto (1, \mathbf{0})$, a *longitudinal* $\epsilon^L \propto (0, \mathbf{k})$, and two *transverse* vectors $\epsilon_{1,2}^T = (0, \mathbf{e}_{1,2}^T)$ with $\epsilon_{1,2}^T \cdot k = \mathbf{e}_{1,2}^T \cdot \mathbf{k} = 0$. Show that the projector on only the transverse states is given by

$$|\epsilon\rangle\langle\epsilon|_{\mu\nu}^{T} = - \sum_{\substack{k\cdot\epsilon=0 \\ \mathbf{k}\cdot\epsilon=0}} \epsilon_\mu^* \epsilon_\nu = g_{\mu\nu} + \frac{k_\mu k_\nu}{\omega^2} \begin{cases} -1 & \text{for } (\mu,\nu) = (0,0) \\ 1 & \text{for } (\mu,\nu) = (i,j) \\ 0 & \text{(otherwise)} \end{cases} \,. \tag{3}$$

Which gauge implements these conditions for massless gauge bosons? Why can we instead, in QED, use Feynman gauge – which instead formally appears to implement Eq. (1) in the propagator – or Landau gauge, which appears to implement Eq. (2) for off-shell photons? How does this argument work in the case of non-Abelian gauge bosons?

d) The fact that the answer in c) is not Lorentz covariant is clearly a disadvantage. This can be fixed by introducing a time-like reference vector n^μ and defining the transverse states as $p \cdot \epsilon = n \cdot \epsilon = 0$. Show that in this way the projection onto transverse polarizations can alternatively be written as

$$|\epsilon\rangle\langle\epsilon|^T_{\mu\nu} = -\sum_{\substack{k\cdot\epsilon=0\\ n\cdot\epsilon=0}} \epsilon^*_\mu \epsilon_\nu = g_{\mu\nu} + \frac{n^2\, k_\mu k_\nu}{(k\cdot n)^2} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n}\,. \tag{4}$$

(Note that this expression is only formally covariant, broken by the fixed vector n^μ.)

Problem 10

Consider the Yang-Mills Lagrangian for a fermion field ('multiplet') ψ charged under some gauge group G:

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \overline{\psi}\,(i\not{D} - m)\,\psi.$$

Now let $G = SU(2)$ and, for concreteness, $\psi = (\nu, e)^T$. Then introduce $W^\pm_\mu \equiv \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)$.

a) What is the explicit form of $\not{D}$, written out as a 2×2 matrix, as a function of the gauge fields $W^\pm$ and A^3? *[Hint: which $SU(2)$ represenation should you use? Write down the corresponding $SU(2)$ generators explicitly!]*

b) Write down all interaction terms between $W^\pm$ and A^3 that are contained in $-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2$.

c) Use the above results to write down the Feynman rules for all vertices involving ν, e, A^3 and/or $W^\pm$.
[While the vector-fermion couplings should be rather straightforward, the vector-vector couplings are more cumbersome to 'read off'. Identify at least which vector couplings exist at all; in a 2nd step, focus on getting the correct vertex rule for at least one of those...]

Problem 11

In this problem we consider the annihilation of fermions into gauge bosons.

a) Let us start with ordinary QED and consider the annihilation of a quark-antiquark pair into two photons. Write down the amplitude. Now consider the high-energy limit of this expression, and compute the (unpolarized) differential cross section $d\sigma/dt$ for $t \to 0$ (i.e. $\theta_{\text{CMS}} \to 0$), expressing the result in Mandelstam variables.
[Hint: Eq. (A.29) in P&S will help a lot!]

b) Now assume that the quarks live in the fundamental representation of $SU(N)$, and consider their annihilation into a pair of gauge bosons associated with that group. Which additional diagram appears, compared to the Abelian case? Compute again $d\sigma/dt$, in the same limit as in a) ! Use this general result to show that for annihilation into a gluon pair (which would be visible as two jets in a collider), you get the QED cross section multiplied by a factor of $\frac{16}{27} Q^{-4} (\alpha_{\text{s}}/\alpha_{\text{em}})^2$.

c) In a non-Abelian theory, unlike the case of QED, one cannot simply replace the gauge boson polarization sum as $\sum \epsilon^*_\mu \epsilon_\nu \to -g_{\mu\nu}$, cf. problem 9c). However, for the specific process studied in b), this would only lead to a mistake for one of the diagrams – which one, and why? Compute the corresponding unphysical contribution to $d\sigma/dt$ in the high-energy limit (but for arbitrary t).