Torsten Bringmann (torsten.bringmann@fys.uio.no)
http://www.uio.no/studier/emner/matnat/fys/FYS5120/v24/

Lecture spring 2024:

Quantum field theory II

Problem sheet 4

Problem 12

Consider a massless scalar field ϕ without self-interactions that is charged under a local $U(1)$ symmetry.

a) Calculate the unpolarized differential cross section $d\sigma/dt$ for the process $\phi\phi^* \to \phi\phi^*$, to lowest order in perturbation theory.

b) How does the result change if ϕ is an N-plet rather than a field with a single complex degree of freedom? (Here, by 'unpolarized' we also mean that we don't know *which* of the multiplet component is in the initial state)

c) Now repeat the same calculation for the case that ϕ is charged under a local $SU(N)$ symmetry, rather than $U(1)$, and that it transforms in the adjoint representation of this group.

Problem 13

Consider a theory containing N real scalar fields:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2 ,$$

where a summation over $i = 1, ..., N$ is understood. Demonstrate explicitly that this theory is symmetric under $SO(N)$, while the vacuum states are only symmetric under $SO(N-1)$. Demonstrate also explicitly that the theory contains $N-1$ massless scalar field, as required by Goldstone's theorem.

Problem 14

Goldstone's theorem can directly be related to Noether's theorem. For any continuous symmetry described by a real parameter α, in particular, the conserved Noether charge is given by

$$Q = \int d^3x\, j_0^N = \int d^3x\, \pi_m \frac{\delta\phi_m}{\delta\alpha} ,$$

where π_m is the conjugate momentum density to the field ϕ_m and a sum over all fields (m) is understood.

a) Show that $[Q, \phi_m] = -i\delta\phi_m/\delta\alpha$ and $[H, Q] = 0$. What is the interpretation of these equalities? In this language, why is spontaneous symmetry breaking characterized by a situation where $Q|\Omega\rangle_{\rm false} = 0$ and $Q|\Omega\rangle_{\rm true} \neq 0$, where $|\Omega\rangle_{\rm false}$ and $|\Omega\rangle_{\rm true}$ are the vacuum states before and after symmetry breaking, respectively?

b) Show that $|\pi(\mathbf{p})\rangle \equiv \int d^3x\, e^{-i\mathbf{px}} j_0^N |\Omega\rangle$ are states with momentum $\mathbf{p}$ and energy $E(\mathbf{p}) + E_0$, where E_0 is the energy of the vacuum state.

c) Show that in the broken phase π describes a massless particle, i.e. $E(\mathbf{p}) = 0$ for $\mathbf{p} \to 0$. Why does this conclusion not hold in the symmetric (false) vacuum state?

Problem 15

Consider again a Yang-Mills theory, now with the gauge group $SU(2)_R$ – i.e. the right-handed part of the fermion field Ψ transforms in the fundamental representation of $SU(2)$ while the left-handed part is not charged. Furthermore, we add a scalar field Φ in the same representation as $\Psi_R = P_R\Psi$:

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\Psi}_L \left(i\not{D}\right) \Psi_L + \bar{\Psi}_R \left(i\not{D}\right) \Psi_R + |D_\mu\Phi|^2 - V(|\Phi|) \\ & + \text{scalar-fermion interaction terms}\,,\end{aligned}$$

where the field strength can be written as $F^a_{\mu\nu} = (D_\mu A_\nu)^a - \partial_\nu A^a_\mu$.

a) How many components does the fermion multiplet Ψ have (and hence Φ), and how many gauge fields are there? Are the 4 covariant derivates appearing in the above expressions identical (state all of them explicitly!)?

b) Write down the missing scalar-fermion interaction terms allowed by gauge-invariance that lead to fermion mass terms if Φ has a non-vanishing vacuum expectation value! *(Hint: Note that the conjugate field $\tilde{\Phi}$, with $\tilde{\Phi}_a \equiv \epsilon_{ab}\Phi^*_b$, transforms in the same way as Φ if the latter is in the fundamental representation – this is a special property of $SU(2)$).* Using the unitary gauge – for which Φ is zero except for one real component – calculate the resulting fermion and gauge boson masses! Discuss differences and similarities of these gauge bosons to those in the standard model!

c) Find all interaction terms between the remaining physical scalar field (after spontaneous symmetry breaking) and the gauge fields, and write down the corresponding vertex rules!