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http://www.uio.no/studier/emner/matnat/fys/FYS5120/v24/

Lecture spring 2024:

Quantum field theory II

Problem sheet 5

$\rightsquigarrow$ These problems are *complementary* to what we covered in the course, basically added on demand, to give you a chance of diving a bit deeper into Majorana fermion mass terms.

Problem 16

A possible way to describe the effect of *charge conjugation* is given by

$$\psi^c \equiv \mathcal{C}\overline{\psi}^T ,$$

where T denotes the transpose and $\mathcal{C}$ is the charge conjugation matrix (acting on a classical field ψ, as opposed to an operator C – as introduced in P&S – which acts on field operators). It satisfies

$$\mathcal{C}^T = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C} .$$

a) In the standard representation, $\mathcal{C}$ is given by $\mathcal{C} = i\gamma_2\gamma_0$. Show explicitly that this choice satisfies the above relations! Show further that the following holds:

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -\left(\gamma^\mu\right)^T$$

b) Using the above relations, show that the charge conjugate of a left-handed spinor is right-handed and vice versa.
[Hint: Calculate first $[\mathcal{C},\gamma^5]$ from the last equation. Use this to show that $\left(\psi_{L,R}\right)^c = P_{R,L}\left(\psi^c\right)$, where $P_{R,L} = (1 \pm \gamma^5)/2$.]

c) We have already discussed that a mass term mixes left- and right-handed fields. For the Dirac case, this can be explicitly seen as:

$$\mathcal{L} \supset -m_D\overline{\psi}\psi = -m_D\overline{\psi}\left(P_R + P_L\right)\psi = -m_D\overline{\psi}_L\psi_R - m_D\overline{\psi}_R\psi_L = -m_D\overline{\psi}_L\psi_R + h.c.$$

In light of the above findings, which two further mass terms can one construct out of the fields ψ_L and ψ_R? These are know as *Majorana mass terms* (a field that satisfies $\psi = \psi^c$ is called a Majorana field). Finally, combine all the three masses (m_D, m_R, m_L) into a 2×2 matrix M to arrive at the following compact form, which combines all possible mass terms of a fermion:

$$\mathcal{L} \supset -\frac{1}{2}\overline{\Psi}M\Psi^c + h.c.$$

(State Ψ and M explicitly!)

Problem 17

In the previous problem, you should have arrived at the following way of combining Dirac and Majorana mass terms:

$$-\mathcal{L}_{DM} = \frac{1}{2}(\bar{\nu}_L, (\bar{\nu}_R)^c) \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} + \text{h.c.} \tag{1}$$

a) The symmetric matrix M here can be diagonalized with an orthogonal matrix T to obtain the two neutrino mass eigenstates m_l and m_h.

$$T^T M T = \begin{pmatrix} m_l & 0 \\ 0 & m_h \end{pmatrix} \tag{2}$$

with

$$T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{3}$$

Show that the the two mass eigenvalues are given by

$$\begin{aligned} m_l &= \frac{1}{2}\left((m_L + m_R) - \sqrt{(m_L - m_R)^2 + 4m_D^2}\right) \\ m_h &= \frac{1}{2}\left((m_L + m_R) + \sqrt{(m_L - m_R)^2 + 4m_D^2}\right) . \end{aligned} \tag{4}$$

b) Show, that the angle θ of the matrix T is given by

$$\tan 2\theta = \frac{2m_D}{m_R - m_L} . \tag{5}$$

c) Assuming that $m_L = 0$ and $m_R \gg m_D$, show that the two mass eigenvalues are given by:

$$\begin{aligned} m_l &= \frac{m_D^2}{m_R} , \\ m_h &= m_R\left(1 + \frac{m_D^2}{m_R^2}\right) \simeq m_R \end{aligned} \tag{6}$$

This is the so called see-saw mechanism for the light neutrino mass m_l: the larger m_R, the smaller is m_l and vice versa.