UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: Relativistic quantum field theory (FYS5120 + FYS9210)

Day of exam: March 20, 2024

Exam hours: 4 hours

This examination paper consists of 5 pages.
(including the title page and the formula collection at the end)

Appendices: none

Permitted materials: 3 A4 pages (two-sided) with own notes.

Make sure that your copy of this examination paper is complete before answering.

Midterm exam

Lecture autumn 2024: Advanced quantum field theory (FYS5120 + FYS9120)

$\rightsquigarrow$ **Carefully read** *all questions before you start to answer them! Note that you don't have to answer the questions in the order presented here, so start with those that you feel most sure about. Problems where slightly more involved/time-consuming calculations are needed are marked with an asterisk (∗); you can fully answer subsequent questions even without actually solving those. Keep your descriptions self-contained, but as short and concise as possible – and generally try to only answer what is actually asked for (just for reference, the complete set of answers comfortably fits on 4.5 LaTeX pages)! Answers given in English are preferred; however, feel free to write in Norwegian/Swedish/Danish if you struggle with formulations!*
Maximal number of available points: **40**.

Good luck!

Problem 1

In the **path integral formalism**, the n-point correlation function for any type of field ϕ can be written as

$$\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle = \frac{\int \mathcal{D}\phi\,\phi(x_1)...\phi(x_n)\exp\left[i\int d^4x\,\mathcal{L}\right]}{\int \mathcal{D}\phi\,\exp\left[i\int d^4x\,\mathcal{L}\right]}\,.$$

a) Re-express this correlation function in terms of the generating functional, and demonstrate explicitly that, to any order in perturbation theory, it is sufficient to know the generating functional for the *free* theory. (*3 points*)

b) As an application of this formalism, we learned that one can simply 'read off' the vertex rules from a given interaction term in the Lagrangian. Draw the Feynman diagram and state the corresponding Feynman rules for each of the following interaction terms:

1. $\mathcal{L} \supset -\lambda\, A^2\bar{\psi}\gamma^5\psi$
2. $\mathcal{L} \supset -\lambda\,|\phi^*\partial_\mu\phi|^2$

Here, ψ describes a fermion, ϕ a scalar and A a vector field. Which mass dimension does the coupling λ have in each of these cases? (*5 points*)

Problem 2

Let us explore the role of **ghosts and unphysical degrees of freedom** for the concrete process of gauge boson production in Yang-Mills theory, $\psi\bar{\psi} \to AA$. For the gauge bosons A we use a basis consisting of two transverse polarization vectors $\epsilon_n^T = (0, \boldsymbol{\epsilon}_n)$ with $n \in \{0, 1\}$ and $\boldsymbol{\epsilon}_n \cdot \mathbf{k} = 0$, where $k = (k^0, \mathbf{k})$ is the momentum of A. Instead of the usual longitudinal and transverse polarizations – i.e. $\epsilon \propto (1, 0)$ and $\epsilon \propto (0, \mathbf{k})$, respectively – we will however use *lightlike polarization vectors*

$$\epsilon^{\pm} \equiv \left(\frac{k^0}{\sqrt{2}\,|\mathbf{k}|}, \pm \frac{\mathbf{k}}{\sqrt{2}\,|\mathbf{k}|} \right). \tag{1}$$

The polarization vectors thus introduced are orthonormal,

$$\epsilon_n^T \cdot \epsilon_m^T = -\delta_{nm}, \qquad \epsilon^+ \cdot \epsilon^- = 1\,, \tag{2}$$

and form a complete basis as, $g_{\mu\nu} = \epsilon_\mu^+ \epsilon_\nu^{-*} + \epsilon_\mu^- \epsilon_\nu^{+*} - \sum_n \epsilon_{n,\mu}^T \epsilon_{n,\nu}^{T\,*}$.

a) Draw the three Feynman diagrams contributing to the process, assigning p_1 (p_2) to the incoming anti-fermion (fermion) momentum, and k_1 and k_2 to the final state gauge boson momenta. Writing the matrix element as

$$i\mathcal{M} \equiv i\mathcal{M}^{\mu\nu}\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2) \equiv i\left(\mathcal{M}_{t+u}^{\mu\nu} + \mathcal{M}_s^{\mu\nu}\right)\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\,,$$

state explicitly the expressions for $\mathcal{M}_{t+u}^{\mu\nu}$ and $\mathcal{M}_s^{\mu\nu}$ that follow from the Feynman rules in Feynman gauge. (Do include *all* color indices in this part of the problem; starting from b), you can leave out the *fermion* color indices for notational ease.) (*4 points*)

b)* Simplify $\mathcal{M}_{t+u}^{\mu\nu}\epsilon_\mu^{+*}(k_1)$ and $\mathcal{M}_s^{\mu\nu}\epsilon_\mu^{+*}(k_1)$ as much as possible. You should find $\mathcal{M}_s^{\mu\nu}\epsilon_\mu^{+*}(k_1) = C^{abc}\left(\bar{v}t^c\gamma_\sigma u\right)\left\{g^{\nu\sigma}(k_2^2 - k_3^2) - k_2^\nu k_2^\sigma\right\}/k_3^2$ and $\mathcal{M}_{t+u}^{\mu\nu}\epsilon_\mu^{+*}(k_1) = C^{abc}\,\bar{v}t^c\gamma^\nu u$, with $k_3 \equiv -k_1 - k_2$ and the term C^{abc} to be determined. (*4 points*)
[*<u>Hints</u>: First write ϵ_μ^+ in a more convenient form than (1). For the $(t+u)$-channel terms you then need to use the Dirac equation to get rid of the denominators; for the s-channel term you can use the fact that the vector current is conserved, $\partial_\mu\bar{\psi}\gamma^\mu\psi = 0$, in order to eliminate some of the k_3 dependence.*]

c) Based on the (explicitly stated) result from b), show that $\mathcal{M}^{\mu\nu}\epsilon_\mu^{+*}(k_1)\epsilon_{n,\nu}^{T\,*}(k_2) = 0$ for any n (and on-shell external vector bosons). Why would one generically expect this? (*3 points*)

d)* Now compute instead $\mathcal{M}_{+-} \equiv \mathcal{M}^{\mu\nu}\epsilon_\mu^{+*}(k_1)\epsilon_\nu^{-*}(k_2)$. Is it problematic that this term does *not* vanish for on-shell momenta k_1 and k_2? (All other combinations of contractions with polarization vectors turn out to vanish, as in c), but you do not need to show this) (*3 points*)

e) Write down the Feynman diagram for ghost production, $f\bar{f} \to c\bar{c}$, and compute the amplitude under the assumption that ghosts can be treated as real particles. (*3 points*)
[*<u>Hint</u>: If you cannot recall the Feynman rules, ghosts enter the Lagrangian as $\mathcal{L} \subset -\bar{c}\partial^\mu D_\mu c$.*]

f) The above results can be summarized as $\left|\mathcal{M}_{f\bar{f}\to c\bar{c}}\right|^2 = \mathcal{M}_{+-}\mathcal{M}^*_{-+} \neq 0$, while $\mathcal{M}^{\mu\nu}\epsilon_\mu{}^*\epsilon_\nu{}^* = 0$ for all other combinations involving polarization vectors ϵ^+ or ϵ^-. Assuming that this holds even for off-shell momenta k_1 and k_2 (which it does), discuss interpretation and consequences of these facts. (*2 points*)

<u>Problem 3</u>

We now turn to aspects of the **standard model** of particle physics (SM).

a) Assume a theory with the same global symmetries and matter content as the SM, including the Higgs field, but where none of the gauge groups is gauged. How many physical Goldstone bosons would there be, and what would be their mass? (*2 points*)

b) Draw a typical diagram that contributes to light-by-light scattering in QED, $\gamma\gamma \to \gamma\gamma$, to lowest order. By dimensional analysis, the cross section at high energies can be written as $\sigma^{\rm QED}_{\gamma\gamma\to\gamma\gamma} = c_0\alpha^n_{\rm em}/s$, where c_0 should be expected to be a constant of order unity and s is the CMS-energy squared; what is the value of n? By which factor does the cross section increase if we allow for all SM fermions in the loop, instead of only electrons? (*4 points*)
[*<u>Hint</u>: you don't have to actually calculate* $|\mathcal{M}|^2$ *to obtain the result.*]

c)* Consider the decay of a Higgs particle to two photons, $h \to \gamma\gamma$. Is this a process that can happen at tree level? Draw the fermion loop diagrams that give the dominant contribution. Now assume that these are the *overall* dominating diagrams giving rise to $h \to \gamma\gamma$ (in the SM, this would be the case if $SU(2)$ was not gauged). Calculate the ratio $\Gamma_{h\to gg}/\Gamma_{h\to\gamma\gamma}$ for that case, i.e. by how much larger the decay rate into a gluon pair is compared to the decay into a photon pair! (*7 points*)

[*<u>Hint</u>: you don't have to fully calculate the individual* $|\mathcal{M}|^2$ *to obtain the result.*]

Formulas that you might find useful:

- Properties of Lie algebra generators

$$(t^a)^\dagger = t^a; \qquad [t^a, t^b] = i f^{abc} t^c; \qquad (t^a_G)_{bc} = -i f^{abc} \tag{3}$$

- Casimir operators

$$\begin{aligned} t^a_r t^a_r &\equiv \mathbf{1} \times C_2(r) = \mathbf{1} \times \begin{cases} \frac{N^2-1}{2N} & \text{for fundamental rep. of SU(N)} \\ N & \text{for adjoint rep. of SU(N)} \end{cases} & (4) \\ \mathrm{Tr}[t^a_r t^b_r] &\equiv \delta^{ab} \times C(r) = \delta^{ab} \times \begin{cases} \frac{1}{2} & \text{for fundamental rep. of SU(N)} \\ N & \text{for adjoint rep. of SU(N)} \end{cases} & (5) \end{aligned}$$

- Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{6}$$

- Vertex rule for three ingoing gauge bosons $A^\mu_a(k_1)$, $A^\nu_b(k_2)$ and $A^\sigma_c(k_3)$:

$$-g f^{abc} \Big\{ g^{\mu\nu}(k_2 - k_1)^\sigma + g^{\nu\sigma}(k_3 - k_2)^\mu + g^{\sigma\mu}(k_1 - k_3)^\nu \Big\} \tag{7}$$